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FURTHER INSIGHT INTO THE STRUCTURE OF BOLD AND TIMID POLICIES

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Abstract

A gambler repeatedly plays a game until either he becomes broke or his fortune becomes equal to or exceeds a target amount. The gambler is allowed to make multiple bets, i.e. stake integral amounts on different alternatives of the game, and more than one bet may win simultaneously. The objective is to determine a strategy that maximizes the probability of attaining the target amount. When all bets have the same gain and alternatives of betting exist such that the relevant plays are feasible, the following results are obtained. For unfavorable games, a bold type of policy is shown to be optimal. A timed type of policy is shown to be best within a restricted class of policies for favorable games. In general, optimal policies contain multiple bets. Based on a numerical example, this is established for roulette also.

DYNAMIC PROGRAMMING

1. Introduction

A gambler desires to increase his fortune from an initial amount $i>0$ to a larger amount N. He repeatedly plays a game until either he becomes broke or his fortune becomes equal to or exceeds the target fortune. The game offers many alternatives of betting. At each play, the gambler can stake any positive integral amount not greater than his current fortune. He is allowed to make multiple bets, that is, he can stake different amounts on different alternatives of the game, and more than one bet may win simultaneously. The gambler wants to determine the optimal way of betting such that he maximizes the probability of attaining his goal.

To define the game, let X be a random variable taking values in the set $S=\{1, 2, \cdots, s\}$ such that $P\{X=s\}=p'_s$. Let $\mathscr{C}$ be a class of subsets of S, that is, $\mathscr{C}=\{A_1, A_2, \cdots, A_n\}$ together with positive integers $k_1, k_2, \cdots, k_n$ such that odds $k_j:1$ are offered on the event $\{X\in A_j\}$, $j=1, 2, \cdots, n$. The game is played by betting non-negative integral amounts $a_1, a_2, \cdots, a_n$ on the events $\{X\in A_1\}, \cdots, \{X\in A_n\}$ respectively. Bet j wins if the event $\{X\in A_j\}$ is realized.

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The corresponding probability is $p_j = \sum_{s \in A_j} p'_s$ and the expected gain per unit amount staked is $g_j = k_j p_j - (1 - p_j)$, $j = 1, 2, \cdots, n$. We speak of a gamble (action) $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$. Let K_i denote the set of all gambles available when the current fortune is i; $K_i = \{\boldsymbol{a} = (a_1, \cdots, a_n): 1 \leqq \sum_{j=1}^{n} a_j \leqq i,\ a_j \geqq 0,$ integer$\}$. A gamble $\boldsymbol{a}$ is called simple if exactly one of the a_j's is not equal to 0. Let K_i^0 denote the set of simple gambles available when the current fortune is i.

A policy is a rule for selecting gambles in a sequence of independent repetitions (plays) of the game. A deterministic policy chooses gambles as a deterministic function of the fortune only. A deterministic policy which always chooses simple gambles will be called a simple policy. Let C_D and C_D^0 denote the classes of deterministic and simple policies respectively.

All alternatives, A_j, with identical odds, k_j, are said to correspond to the same option of betting. We say that an option with odds k is available if there is at least one alternative with odds k. A bet on an option with odds k is a bet on a single alternative with odds k.

In Section 2, we formulate this problem as the optimal first-passage problem in Derman (1970), p. 28. Immediate consequences of this formulation are: (i) Only deterministic policies need to be considered as candidates for the optimal policy. (ii) Optimality can be proven by establishing a set of optimality equations. (iii) Policy improvement, linear programming and successive approximations algorithms can be used to obtain optimal policies.

In Sections 3 and 4, it is assumed that all alternatives of the game have the same expected gain per unit amount staked, that is, $g_j = g$ for every j. Equivalently, for $C = g + 1$, it is assumed that

$$p_j = \frac{C}{k_j + 1}, \qquad j = 1, 2, \cdots, n. \tag{1}$$

We say that the game is favorable if $C > 1$, unfavorable if $C < 1$ and fair if $C = 1$.

From (1), all alternatives corresponding to the same option of betting have the same odds and probability of winning. For convenience, we refer to the probability of winning of any alternative with odds k as $\bar{p}_k$. It is given by

$$\bar{p}_k = \frac{C}{k + 1}. \tag{2}$$

It is expedient to describe simple policies in terms of bets on options rather than bets on particular alternatives of the game. Two simple policies are of interest: (i) The first policy always employs a bet of 1 unit on the option with odds 1. We call it the timid policy. (ii) The second policy, when the current fortune is i, employs a bet of 1 unit on the option with odds $N - i$. We call it the bold policy. Note that these policies are uniquely defined up to the option of betting only.

The results of Sections 3 and 4 are the following: (i) For unfavorable games, under the additional assumption that options with odds k are available for all $k=1, 2, \cdots, N-1$, it is shown that the bold policy is the unique optimal policy. (ii) If an option with odds 1 is available then it is shown that the timid policy is the best policy within the class of simple policies for favorable games. We give an example which shows that the timid policy is not optimal if multiple bets are allowed.

In Section 5 we consider roulette. We give a numerical example for which the best simple policy (which is obtained using policy improvement) is not overall optimal. Thus, it follows that optimal policies, in general, contain multiple bets and therefore are difficult to characterize.

The problem of optimal gambling in roulette has been considered by Smith (1967) and Dubins (1968) under the assumption that the gambler is allowed to bet any fraction of his fortune on alternatives of the form $\{X=j\}$, $j \in S$ only. This class of policies has the property that if a bet on an alternative $\{X=j_1\}$ wins, all other bets will lose. This property is used to show that a bold policy is optimal. In our gambling model, bets on more than one alternative may win simultaneously; for example consider bets on alternatives of the form $A_1 = \{j_1, j_2\}$, $A_2=\{j_2, j_3\}$, etc. This general class of gambling policies has been considered by Breiman (1961). Breiman also allows the gambler to bet any fraction of his fortune and considers only those games in which there exists a policy such that the gambler's fortune almost surely increases without bound (favorable games). He considers two criteria. The first is to minimize the expected number of trials needed to reach or exceed a target fortune. The second is to maximize the fortune after a fixed number of trials. In either case, he obtains characterizations for optimal policies.

Another criterion is to maximize the gambler's playing time till he becomes broke. Freedman (1967), Molenaar and Van Der Velde (1967) and Ross (1974) have shown that a timid policy is optimal in various favorable situations.

2. Preliminaries

Let $\Delta_j = \Delta_j(X)$ be the indicator function of the event $\{X \in A_j\}$, $1 \leqq j \leqq n$. We define the random variable $\mathbf{\Delta} = (\Delta_1, \cdots, \Delta_n)$, then $D = \{0, 1\}^n$ is the range of $\mathbf{\Delta}$. For any $\boldsymbol{\delta} \in D$, let $p(\boldsymbol{\delta}) = p(\mathbf{\Delta} = \boldsymbol{\delta})$. If gamble $\boldsymbol{a}$ is placed when the current fortune is i, $1 \leqq i \leqq N-1$, then with probability $p(\boldsymbol{\delta})$ the new fortune will be

$$j(i, \boldsymbol{a}, \boldsymbol{\delta}) = i + \sum_{j=1}^{n} \delta_j a_j k_j - \sum_{j=1}^{n} (1-\delta_j) a_j = i + \sum_{j=1}^{n} \delta_j a_j (k_j + 1) - |\boldsymbol{a}|. \tag{3}$$

In (3), $|\boldsymbol{a}| = \sum_{j=1}^{n} a_j$ is the total amount gambled; $|\boldsymbol{a}| \leqq i$. Finally, let $l(i, \boldsymbol{a}) = i - |\boldsymbol{a}| + 1$.

Remark 1.

(i) The $p(\boldsymbol{\delta})$, $\boldsymbol{\delta} \in D$, can be obtained as follows. Define $B(\boldsymbol{\delta}) = (\bigcap_{j:\delta_j=1} A_j) \cap (\bigcup_{i:\delta_i=0} A_i)^C$.

Then

$$p(\boldsymbol{\delta}) = P(X \in B(\boldsymbol{\delta})) = \sum_{s \in B(\boldsymbol{\delta})} p'_s.$$

(ii)

$$\sum_{\boldsymbol{\delta} \in D} p(\boldsymbol{\delta}) = 1 \tag{4}$$

$$\sum_{\boldsymbol{\delta} \in D:\delta_j=1} p(\boldsymbol{\delta}) = \sum_{\boldsymbol{\delta} \in D} \delta_j p(\boldsymbol{\delta}) = p_j \qquad 1 \leqq j \leqq n. \tag{5}$$

Using standard Markov decision theory arguments, Derman (1970), a deterministic policy R^* is optimal if and only if

$$f^{R^*}(i) = \max_{\boldsymbol{a} \in K_i} \left\{ \sum_{\boldsymbol{\delta} \in D} p(\boldsymbol{\delta}) f^{R^*}(j(i, \boldsymbol{a}, \boldsymbol{\delta})) \right\} \qquad 1 \leqq i \leqq N-1 \tag{6}$$

where $f^{R^*}(i)$ denotes the first-passage probability to the set of states $\{N, N+1, \cdots\}$ given that the initial state is i, $1 \leqq i \leqq N-1$, and policy R^* is employed.

A simple policy R^0 is optimal within the class of simple policies if the $f^{R^0}(i)$ satisfy (6) with K_i replaced by K_i^0, $1 \leqq i \leqq N-1$. Let

$$d(i) = f(i) - f(i-1) \qquad 1 \leqq i \leqq N-1 \tag{7}$$

and

$$T(i, \boldsymbol{a}) = \sum_{\boldsymbol{\delta} \in D} p(\boldsymbol{\delta}) f(j(i, \boldsymbol{a}, \boldsymbol{\delta})), \qquad 1 \leqq i \leqq N-1, \boldsymbol{a} \in K_i.$$

Here, and in subsequent sections, the superscript denoting the policy will be omitted for notational simplicity. In the following lemma, the expressions $T(i, \boldsymbol{a})$ which appear in (6), are shown to be equal to a weighted sum of the differences $d(i)$, with the weights non-increasing in i. This result is true for all deterministic policies and all feasible gambles.

Lemma 1. Assume that condition (1) holds. Then for any i, $1 \leqq i \leqq N-1$ and any $\boldsymbol{a}$, $\boldsymbol{a} \in K_i$, there exist constants $b_r = b_r(i, \boldsymbol{a})$, $l(i, \boldsymbol{a}) \leqq r \leqq N$, such that

$$b_{l(i,\boldsymbol{a})} \geqq b_{l(i,\boldsymbol{a})+1} \geqq \cdots \geqq b_N \geqq 0, \quad \sum_{r=l(i,\boldsymbol{a})}^{N} b_r \leqq 1$$

and

$$T(i, \boldsymbol{a}) - f(i - |\boldsymbol{a}|) = C\,|\boldsymbol{a}| \sum_{r=l(i,a)}^{N} b_r d(r).$$

Proof. Using (4) and (7)

$$T(i, \boldsymbol{a}) - f(i - |\boldsymbol{a}|) = \sum_{\boldsymbol{\delta} \in D} p(\boldsymbol{\delta})(f(j(i, \boldsymbol{a}, \boldsymbol{\delta})) - f(i - |\boldsymbol{a}|)) = \sum_{\boldsymbol{\delta} \in D} p(\boldsymbol{\delta}) \sum_{r=l(i,\boldsymbol{a})}^{j(i,\boldsymbol{a},\boldsymbol{\delta})} d(r). \tag{8}$$

To interchange the order of summation in (8), define

(9) $$D_r = D_r(i, \boldsymbol{a}) = \{\boldsymbol{\delta} \in D : j(i, \boldsymbol{\delta}, \boldsymbol{a}) \geqq r\}, \qquad r = l(i, \boldsymbol{a}), \cdots, N.$$

Note that $D_N \subseteq D_{N-1} \subseteq \cdots \subseteq D_{l(i,\boldsymbol{a})}$. Let

$$i_{\boldsymbol{a}} = \max_{\boldsymbol{\delta} \in D} j(i, \boldsymbol{a}, \boldsymbol{\delta}).$$

Now (8) can be written as

$$T(i, \boldsymbol{a}) - f(i - |\boldsymbol{a}|) = \sum_{r=l(i,\boldsymbol{a})}^{i_{\boldsymbol{a}}} d(r) \sum_{\boldsymbol{\delta} \in D_r} p(\boldsymbol{\delta}).$$

If $i_{\boldsymbol{a}} < N$, then $D_r = \varnothing$ and $\sum_{\boldsymbol{\delta} \in D_r} p(\boldsymbol{\delta}) = 0$ for $r = i_{\boldsymbol{a}} + 1, \cdots, N$. Also $d(r) = 0$ for $r > N$. Hence

(10) $$T(i, \boldsymbol{a}) - f(i - |\boldsymbol{a}|) = C\,|\boldsymbol{a}| \sum_{r=l(i,\boldsymbol{a})}^{N} d(r) b_r$$

where $b_r = (1/C\,|\boldsymbol{a}|) \sum_{\boldsymbol{\delta} \in D_r} p(\boldsymbol{\delta})$. It follows from (9) that

$$b_{l(i,\boldsymbol{a})} \geqq b_{l(i,\boldsymbol{a})+1} \geqq \cdots \geqq b_N \geqq 0.$$

To show that the sum of the b_r's is less than or equal to 1, substitute $d(r) = 1$ for every r in (8) and (10). Since all terms containing $d(r)$, $r > N$, are excluded in (10), it follows that

$$\begin{aligned}
\sum_{r=l(i,\boldsymbol{a})}^{N} b_r &\leqq \frac{1}{C\,|\boldsymbol{a}|} \sum_{\boldsymbol{\delta} \in D} p(\boldsymbol{\delta}) \sum_{r=l(i,\boldsymbol{a})}^{j(i,\boldsymbol{a},\boldsymbol{\delta})} 1 \\
&= \frac{1}{C\,|\boldsymbol{a}|} \sum_{\boldsymbol{\delta} \in D} p(\boldsymbol{\delta}) \sum_{j=1}^{n} \delta_j (k_j + 1) a_j \\
&= \frac{1}{C\,|\boldsymbol{a}|} \sum_{j=1}^{n} a_j (k_j + 1) \sum_{\boldsymbol{\delta} \in D} \delta_j p(\boldsymbol{\delta}) = \frac{1}{C\,|\boldsymbol{a}|} \sum_{j=1}^{n} a_j (k_j + 1) p_j \\
&= \frac{1}{C\,|\boldsymbol{a}|} \sum_{j=1}^{n} a_j C = 1
\end{aligned}$$

where the first equation follows from (3), the third from (5) and the last two equations from (2) and the definition of $|\boldsymbol{a}|$.

The proof of Lemma 1 is complete.

The next lemma will be used in the following section. The proof is simple and is omitted.

Lemma 2. If $d_m > d_{m-1} > \cdots > d_1 > 0$; $0 \leqq b_m \leqq b_{m-1} \leqq \cdots \leqq b_1$ and $\sum_{r=1}^{m} b_r \leqq 1$ then

$$\frac{1}{m} \sum_{r=1}^{m} d_r \geqq \sum_{r=1}^{m} b_r d_r$$

with equality holding if and only if $b_r = 1/m$ for $r = 1, 2, \cdots, m$.

3. The unfavorable case $(C<1)$

The bold policy has been defined as the simple policy which in state i bets one unit on the option with odds $N-i$, $1 \leqq i \leqq N-1$. Here we assume that options with odds $1, 2, \cdots, N-1$ exist. For this policy, the first-passage probabilities satisfy

$$f(i)=\bar{p}_{N-i}f(N)+(1-\bar{p}_{N-i})f(i-1), \quad i=1,2,\cdots,N-1. \tag{11}$$

The unique solution to the system of equations given above is

$$f(i)=1-\prod_{j=1}^{i}(1-\bar{p}_{N-j}), \quad i=1,2,\cdots,N-1. \tag{12}$$

The next lemma follows from (11) and (12).

Lemma 3.

(i) $d(i+1)>d(i), \qquad i=1,2,\cdots,N-1$

(ii) $f(i)-f(i-m)\geqq \dfrac{Cm}{N-i+m}\displaystyle\sum_{j=i-m+1}^{N} d(j), \qquad i=1,\cdots,N-1;\ m=1,2,\cdots,i,$

with equality holding if and only if $m=1$.

Proof. (i) The first part follows from (12) and the fact that $C<1$.

(ii) From (11),

$$f(i-m+1)=\bar{p}_{N-i+m-1}f(N)+(1-\bar{p}_{N-i+m-1})f(i-m)$$

equivalently, using (2) and (7),

$$f(i-m+1)-f(i-m)=\frac{C}{N-i+m}\sum_{j=i-m+1}^{N} d(j).$$

By repeated application of (i), we obtain that

$$f(i)-f(i-m)\geqq m[f(i-m+1)-f(i-m)]$$

with equality holding if and only if $m=1$, and the result follows.

Remark 2. Substituting $m=1$ in Lemma 3 (ii), it follows that in state i the b_r's corresponding to the gamble prescribed by the bold policy are all equal to $1/(N-i+1)$. It is easy to see that a simple gamble in state i of 1 unit on any option with odds not equal to $N-i$ does not have this property.

Theorem 1. If options with odds $1, 2, \cdots, N-1$ are available, then the bold policy is the unique (up to bets on the same option) optimal policy for the unfavorable case.

Proof. It suffices to show that

$$f(i) > T(i, \boldsymbol{a}), \qquad i = 1, 2, \cdots, N-1$$

for all alternative gambles $\boldsymbol{a} \in K_i$ such that $\boldsymbol{a}$ is not equal to a gamble prescribed by the bold policy. From Lemma 1 we have that

$$T(i, \boldsymbol{a}) - f(i - |\boldsymbol{a}|) = C\,|\boldsymbol{a}| \sum_{r=l(i,\boldsymbol{a})}^{N} b_r d(r).$$

From Lemma 3 (ii), for $m = |\boldsymbol{a}|$, we have that

$$f(i) - f(i - |\boldsymbol{a}|) \geqq C\,|\boldsymbol{a}| \frac{1}{N - i + |\boldsymbol{a}|} \sum_{r=l(i,\boldsymbol{a})}^{N} d(r)$$

with equality holding if and only if $|\boldsymbol{a}| = 1$. Finally, from Lemma 2,

$$\frac{1}{N - i + |\boldsymbol{a}|} \sum_{r=l(i,\boldsymbol{a})}^{N} d(r) \geqq \sum_{r=l(i,\boldsymbol{a})}^{N} b_r d(r)$$

with equality holding if and only if $b_r = 1/(N - i + |\boldsymbol{a}|)$ for every r. The proof is complete if we note that for $|\boldsymbol{a}| = 1$, the only gambles for which $b_r = 1/(N - i + |\boldsymbol{a}|)$ for every r are the ones prescribed by the bold policy.

4. The favorable case ($C > 1$) and the fair case ($C = 1$)

If an option with odds 1 is available, the timid policy is feasible. Under this policy the first-passage probabilities are given by

$$f(i) = \frac{1 - \rho^i}{1 - \rho^N}, \qquad i = 1, 2, \cdots, N-1 \tag{13}$$

where

$$\rho = \frac{1 - \bar{p}_1}{\bar{p}_1} = \frac{2 - C}{C}. \tag{14}$$

A simple gamble consists of one non-zero bet, say bet 1. The only possible outcomes are whether this bet wins or loses. Thus, the timid policy is the best policy within the class of simple policies if

$$f(i) \geqq \bar{p}_k f(i + ak) + (1 - \bar{p}_k) f(i - a), \qquad i = 1, 2, \cdots, N-1; \quad a = 1, 2, \cdots, i; \quad \text{for every } k. \tag{15}$$

These inequalities are a special case of (7).

Theorem 2. If the option with odds 1 is available, the timid policy is the unique (up to bets on the same option) optimal policy within the class of simple policies for the favorable case.

Proof. Equations (13) and (14) imply that (15) can be written as

$$(k+1)(1-\rho^{a})-2(1-\rho^{a(k+1)})/(1+\rho)\geqq 0.$$

Now, in the favorable case, $C>1$ thus $\rho<1$ and the preceding inequality is easily confirmed to hold for $k=1$. The other cases follow as the left-hand side is increasing in k.

The following example shows that when multiple bets are allowed the timid policy is not optimal in general.

Assume that alternatives A_1, A_2 have odds $k_1=k_2=1$ and that $A_1\cup A_2=S$. Since the game is favorable, $A_1\cap A_2\neq\varnothing$ and it therefore follows that a unit bet on A_1 and A_2 cannot lead to a loss and, with positive probability, may lead to a profit.

Remark 3. For the fair case, all policies which prescribe gambles $\boldsymbol{a}$ in state i such that $j(i, \boldsymbol{a}, \boldsymbol{\delta})\leqq N$ for every $\boldsymbol{\delta}\in D$, result in identical first-passage probabilities. To see this, notice that for the timid policy

$$f(i)=\frac{i}{N},\qquad i=1,2,\cdots,N-1,$$

as obtained from the gambler's ruin model. It follows that

$$d(i)=d(i-1),\qquad i=2,3,\cdots,N$$

and the optimality equations (6) are satisfied as equalities for every gamble $\boldsymbol{a}\in K_i^*$ where

$$K_i^*=\{\boldsymbol{a}\in K_i : j(i, \boldsymbol{a}, \boldsymbol{\delta})\leqq N \text{ for every } \boldsymbol{\delta}\in D\}.$$

5. Roulette

The rules for playing roulette are given in Scarne (1961). Briefly, using the notation of our gambling model, $S=\{00, 0, 1, \cdots, 36\}$ with $p'_s=1/38$ for every $s\in S$. There are nine options of betting available, as summarized in Table 1. Only the fifth option does not satisfy Equations (2).

The odds corresponding to the ninth option cannot be written as $k:1$ for some positive integer k. This introduces the possibility of winning non-integral amounts. Therefore, we stipulate that only bets of even amounts can be placed on the alternatives of the game corresponding to the ninth option.

We have used the policy improvement algorithm to solve numerically for the best simple policy. The solution for $N=10$ is given in Table 2 and will be used to show that the optimal policy, in general, consists of multiple bets. In Table 2, we also list the first-passage probabilities corresponding to the timid and the bold policies with $C=36/38$. Note that the timid policy corresponds to bets on red or black in roulette. The bold policy is not feasible except for $N\leqq 3$.

TABLE 1
Options of betting in roulette

option	numbers bet on	probability of winning	odds	expected return per unit amount bet
1	1	1/38	35 : 1	36/38
2	2	2/38	17 : 1	36/38
3	3	3/38	11 : 1	36/38
4	4	4/38	8 : 1	36/38
5	5	5/38	6 : 1	35/38
6	6	6/38	5 : 1	36/38
7	12	12/38	2 : 1	36/38
8	18	18/38	1 : 1	36/38
9	24	24/38	$\frac{1}{2}$: 1	36/38

However, it provides an upper bound for the first-passage probabilities since it assumes that options with odds $1, 2, \cdots, N-1$ exist each with an expected return per unit amount bet, C, equal to 36/38.

The following example shows that the best simple policy obtained above is not overall optimal.

Let $A_1 = \{1, 2, \cdots, 6\}$ and $A_2 = \{1, 2, \cdots, 12\}$. From Table 1, since bet 1 is a bet on six numbers, $k_1 = 5$, similarly, $k_2 = 2$. In state 3, consider the gamble $(1, 1)$ where, for simplicity, we assume that $C = \{A_1, A_2\}$. Let $\boldsymbol{\delta}_1 = (1, 1)$, $\boldsymbol{\delta}_2 = (1, 0)$, $\boldsymbol{\delta}_3 = (0, 1)$ and $\boldsymbol{\delta}_4 = (0, 0)$. Then, $p(\boldsymbol{\delta}_1) = 6/38$, $p(\boldsymbol{\delta}_2) = 0$, $p(\boldsymbol{\delta}_3) = 6/38$ and $p(\boldsymbol{\delta}_4) = 26/38$. The corresponding optimality inequality is

$$f(3) \geqq 6/38 f(10) + 6/38 f(4) + 26/38 f(1).$$

Substitution of the values from Table 2 shows that this inequality is violated.

TABLE 2
The best simple policy for $N = 10$ and the first-passage probabilities corresponding to the timid and the bold policies.

state i	best simple policy: amount staked	best simple policy: bet on odds	$f(i)$	timid policy	bold policy
1	1	8 : 1	0·09288	0·05948	0·09474
2	1	8 : 1	0·18837	0·12557	0·19003
3	1	5 : 1	0·28123	0·19901	0·28595
4	3	2 : 1	0·37934	0·28060	0·38258
5	1	5 : 1	0·47734	0·37126	0·48007
6	2	2 : 1	0·57534	0·47200	0·57858
7	3	1 : 1	0·67334	0·58392	0·67839
8	2	1 : 1	0·77649	0·70829	0·77996
9	1	1 : 1	0.88236	0.84647	0.88419

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