Optimal Repair of a 2-Component Series-System with Partially Repairable Components

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Reader Aids—
Purpose: Present a derivation.
Special math needed for explanations: Probability, dynamic programming
Special math needed to use results: Same
Results useful to: Maintenance theoreticians

Abstract—A 2-component series system is maintained by one repairman. The up-times of the components are *s*-independent r.v.'s with exponential distributions. The time required to repair a failed component is the sum of a number of *s*-independent, exponential r.v.'s. Components can be partially repaired, and a working component can fail even while the system as a whole is not functioning. The analysis finds repairman allocation policies which maximize the system availability. Under the assumption that it is permissible to reassign the repairman instantaneously among failed components, the explicit form of optimal policies is obtained. And, the optimal policies are characterized when the time for such a reassignment is allowed to be an exponential r.v.

1. INTRODUCTION

Consider a 2-component series (1-out-of-2:F) system maintained by one repairman. The up-times of the components are *s*-independent r.v.'s with exponential distributions. Repair of a failed component consists of several stages. The time required to complete a stage is an *s*-independent exponential r.v. A component is partially repaired if some, but not all, stages of repair have been completed. The component begins functioning as soon as all stages of repair are complete, even if the system as a whole is failed. The problem is to decide which failed component should be under repair so that the set of all these decisions in time (ie, the maintenance policy) maximizes the average *s*-expected system operation time, ie, the availability.

The series system has been studied under the assumption that the repair of each component is of exponential duration and is *s*-independent of the state of the system, and that it is allowed to reassign the repairman instantaneously. Smith [9, 10] has shown that the policy π^*, which assigns the repairman to the failed component with the smallest failure rate, is optimal for: i) a 2-component system, ii) an *n*-component system within a class of "list policies", and iii) in the limit when the failure rates tend uniformly to zero. Derman et al. [3] established the optimality of π^* for the case in which all repair rates are equal. Finally, Katehakis & Derman [7] proved that π^* is optimal for all values of the repair and failure rates.

A common (and the simplest) way to maintain such a system is to assign the repairman to a failed component, wait till the component is fully repaired and then let the repairman idle until the next component failure occurs. If the other component, however, is already failed then the repairman goes to work on it immediately. This maintenance policy does not require any special instructions for the repairman.

This policy need not be the best way to maintain the system and it might be beneficial to reassign the repairman from one failed component, before it is completely repaired, to the other failed component. However, such a reassignment would require special instructions for the repairman and a time penalty could be assessed each time it is done. Two cases are studied.

1. There is no penalty, that is, reassignments are instantaneous. The optimal policy is first to complete all but one stages of repair on both components, reassigning the repairman as necessary, and then to complete the last stage of repair in a predetermined order based only on the values of the repair and failure rates.

2. A penalty of a random amount of time, which is exponentially distributed with rate λ_s, is assessed. If λ_s is greater than a certain constant α_1 then the reassignment policy in case 1 should be used. On the other hand, if λ_s is less than another constant α_2 then the repairman should not be reassigned at all. The values of α_1, α_2 are obtained in terms of the failure and repair rates.

2. INSTANTANEOUS REASSIGNMENT MODEL

Assumptions:

1. The up-time of component i is an exponentially distributed r.v. with rate μ_i, $i = 1, 2$.
2. Component i requires k_i stages of repair. The time to complete stage j is an exponentially distributed r.v. with rate λ_{ij}.
3. The repairman may be reassigned instantaneously from one failed component to another.
4. The repairman is not allowed to be idle while a component is failed.

Notation:

k_i number of stages of repair required for component i.

λ_{ij} rate of the exponentially distributed repair time of state j of component i.

μ_i rate of the exponentially distributed up-time of component i.

N (n_1, n_2) state of the system; $n_i = j$ if and only if component i has j stages of repair completed

S $\{N = (n_1, n_2): n_1 = 0, 1, ..., k_1; n_2 = 0, 1, ..., k_2\}$ set of all possible states

K (k_1, k_2) the only state in which the system functions.

$C_0(N)$ $\{i: n_i < k_i\}$ set of failed components

$C_1(N)$ $\{i: n_i = k_i\}$ set of working components

$(1_j, N)$ $\begin{cases}(n_1 + 1, n_2), & \text{for } j = 1\\(n_1, n_2 + 1), & \text{for } j = 2\end{cases}$

$(0_j, N)$ $\begin{cases}(0, n_2), & \text{for } j = 1\\(n_1, 0), & \text{for } j = 2\end{cases}$

R_i $\sum_{j=1}^{k_i} (1/\lambda_{ij})$

$\mu(N)$ $\sum_{j\in C_1(N)} \mu_j$

Π finite set of deterministic policies

$T_\pi(N)$ s-expected first passage time from state N to state K for a policy $\pi \in \Pi$

$\pi(N)$ the component the repairman is assigned to by policy π when the system is in state N

$\lambda(\pi(N))$ λ_{i,n_i+1} for $\pi(N) = i$

Conditions for Optimality

A policy $\pi^* \in \Pi$ is optimal with respect to the maximum availability criterion if and only if:

$$T_{\pi^*}(N) \le T_\pi(N), \text{ for every } N \in S, \text{ for every } \pi \in \Pi. \quad (1)$$

This statement is proved in [8, 9]. This proof is outlined below.

When a deterministic policy is used, the time evolution of the system state can be described by a continuous time, finite state, irreducible Markov chain. Furthermore, the s-expected average system operation time is then equal to the steady-state probability of the system's being in state K. Returns to state K generate an alternating renewal process with the property that the sojourn time in state K and the transition probabilities to other states are independent of the policy. Thus, maximizing the steady-state probability of the system's being in state K is equivalent to minimizing the s-expected first passage times to state K over all possible initial states.

By conditioning on the first transition out of state N, the $T_\pi(N)$'s are the unique solution to the system of linear equations (2):

$$T_\pi(N) = \frac{1}{\lambda(\pi(N)) + \mu(N)} \left[1 + \lambda(\pi(N))T_\pi(1_{\pi(N)}, N) + \sum_{j\in C_1(N)} \mu_j T_\pi(0_j, N)\right]$$
$$N \in S - \{K\} \quad (2)$$
$$T_\pi(K) = 0$$

It is a standard result of Markov decision processes [2, 8], that a policy π^* is optimal if and only if the associated s-expected first passage times $T_{\pi^*}(N)$ satisfy (3):

$$T_{\pi^*}(N) = \min_{\alpha\in C_0(N)} \left[\frac{1}{\lambda(\alpha) + \mu(N)} \left\{1 + \lambda(\alpha)T_{\pi^*}(1_\alpha, N) + \sum_{j\in C_1(N)} \mu_j T_{\pi^*}(0_j, N)\right\}\right]$$
$$N \in S - \{K\} \quad (3)$$

It follows that π^* is optimal if and only if the inequalities (4) can be established.

$$T_{\pi^*}(N) \le \frac{1}{\lambda(\alpha) + \mu(N)} \left[1 + \lambda(\alpha)T_{\pi^*}(1_\alpha, N) + \sum_{j\in C_1(N)} \mu_j T_{\pi^*}(0_j, N)\right]$$
$$N \in S - \{K\}, \alpha \in C_0(N) - \{\pi^*(N)\}. \quad (4)$$

Furthermore, π^* is the unique optimal policy if inequalities (4) are strict.

Optimal Reassignment Policies

Ref. [6] shows that any deterministic policy π_1 which satisfies (5) and (6) also satisfies inequalities (4) and hence is optimal with respect to the maximum availability criterion. A proof is given in [12].

$$\pi_1(N) = \begin{cases}1, & \text{for } N \in S: n_1 < k_1 - 1, n_2 = k_2 - 1\\2, & \text{for } N \in S: n_1 = k_1 - 1, n_2 < k_2 - 1\end{cases} \quad (5)$$

$$\pi_1(k_1 - 1, k_2 - 1) = \begin{cases}1, & \text{for } \mu_1\lambda_{1k_1}R_1 \le \mu_2\lambda_{2k_2}R_2\\2, & \text{otherwise}\end{cases} \quad (6)$$

3. REASSIGNMENT PENALTY MODEL

Assumptions

1. The up-time of component i is an exponentially distributed r.v. with rate μ_i, $i = 1, 2$.

2. Both components require k stages of repair. The time to complete any stage of repair of component i is exponentially distributed with rate λ_i.

3. Any reassignment of the repairman, other than upon completion of all stages of repair on a component, incurs a time penalty. A random amount of time exponentially distributed with rate λ_s is required for such a reassignment.

4. The repairman is not idle while a component is failed.

Notation

k number of stages of repair required for each component

λ_i rate of the exponentially distributed repair time of any stage of repair of component i

μ_i rate of the exponentially distributed up-time of component i

N (n_1, n_2, r) state of the system; $n_i = j$ if and only if component i has j stages of repair completed; r denotes the component currently under repair ($r = 0$ if both components are working)

S set of all possible states

K $(k, k, 0)$ the only state in which the system functions.

π deterministic policy

$\pi(N)$ the component the repairman is assigned to by policy π when the system is in state N.

$\lambda(\pi(N))$ λ_r, for $\pi(n_1, n_2, r) = r$; λ_s, otherwise.

The rest of the terms are defined as in section 2 but with the new state vector $N = (n_1, n_2, r)$.

Conditions for Optimality

As in section 2, a policy π^* is optimal if and only if it satisfies (3). Optimality can be shown by establishing inequalities (4).

States for which the Option of Reassigning the Repairman is Inferior

Consider, for example, the system in state $(n_1, n_2, 1)$ with $n_1 < k - 1$, $n_2 \leq k - 1$. The options available in this state are: to continue repairing component 1, or to reassign the repairman to component 2 and to incur the penalty. It is clearly inferior to reassign the repairman unnecessarily. Thus, if the reassignment option is chosen, the repairman must continue to repair component 2 until either i) component 2 is functional, or ii) $k - 1$ stages of repair are complete and the repairman is reassigned back to component 1. For case i, the repairman resumes repair of component 1. However, component 2 is working now and can fail during the repair of component 1. Contrast this with the alternative of reassigning the repairman from component 1 to component 2 only when $k - 1$ stages of repair have been completed on component 1. When the repairman returns to component 1, after completing the repair on component 2, only one stage of repair needs to be completed. Hence, the chances of component 2 failing in the meantime are smaller than for case i); therefor case i is clearly an inferior repair policy. This alternative of reassigning only when $k - 1$ stages of repair are complete on component 1 is also at least as good as case ii.

Using similar arguments as above for other states, the option of reassigning the repairman needs to be considered only in states $(k - 1, 0, 1)$, $(0, k - 1, 2)$, $(k - 1, k - 1, 1)$, $(k - 1, k - 1, 2)$.

As a consequence of not reassigning the repairman unless at least $k - 1$ stages of repair are complete, states (n_1, n_2, r), for $0 < n_1 < k - 1, 0 < n_2 < k - 1, r = 1, 2$ are never encountered.

Optimality of the Reassignment Policy

Without loss of generality assume $\mu_1 \leq \mu_2$. Define the reassignment policy π_2 as:

$$\pi_2(n_1, n_2, 1) = 1, \text{ for } n_1 = 0, 1, \ldots, k\text{-}2;\ n_2 = 0, k\text{-}1, k.$$

$$\pi_2(n_1, n_2, 2) = 2, \text{ for } n_1 = 0, k\text{-}1, k;\ n_2 = 0, 1, \ldots, k\text{-}2.$$

$$\pi_2(k\text{-}1, k\text{-}1, 2) = \pi_2(0, k\text{-}1, 2) = \pi_2(k\text{-}1, k\text{-}1, 1) = 1$$

$$\pi_2(k\text{-}1, 0, 1) = 2$$

π_2 gives priority to component 1 since $\mu_1 \leq \mu_2$. If reassignment has been instantaneous π_2 would have been the optimal policy. π_2 is the unique optimal policy if $\lambda_s \in (\alpha_1, \infty)$ for α_1 defined by (7). For the proof refer to [12].

$$\alpha_1 \equiv \max \left\{ \frac{\lambda_1\lambda_2 + \lambda_1\mu_1 - \lambda_2\mu_2}{k(\mu_2 - \mu_1)}, \frac{2\lambda_1\lambda_2 - \lambda_1\mu_1 c}{\mu_1 k c} \right\}$$

$$c \equiv \sum_{i=1}^{k-1} [\lambda_2/(\lambda_2 + \mu_1)]^i \tag{7}$$

Optimality of the Non-reassignment Policy

Define the polity π_3 as the policy which never reassigns the repairman. π_3 is identical to π_2 except that:

$$\pi_3(k\text{-}1, k\text{-}1, 2) = \pi_3(0, k\text{-}1, 2) = 2,$$

$$\pi_3(k\text{-}1, 0, 1) = 1.$$

π_3 is the unique optimal policy if $\lambda_s \in (0, \alpha_2)$ for α_2 defined by (8)-(10). For the proof refer to [12].

$$\alpha_2 \equiv \left[\max \left\{ \frac{\lambda_1}{\lambda_1 + \mu_2} d - \frac{k}{\lambda_2}, \frac{\lambda_2}{\lambda_2 + \mu_1} e - \frac{k}{\lambda_1} \right\} \right]^{-1} \tag{8}$$

$$d \equiv \frac{k}{a + b - ab} \left[\frac{1 - a}{\lambda_1} + \frac{1}{\lambda_2} \right]$$

$$e \equiv \frac{k}{a + b - ab} \left[\frac{1}{\lambda_2} + \frac{1 - b}{\lambda_2} \right] \tag{9}$$

$$a \equiv [\lambda_2/(\lambda_2 + \mu_1)]^k$$

$$b \equiv [\lambda_1/(\lambda_1 + \mu_2)]^k \tag{10}$$

4. EXAMPLES

Example 1: $k = 2$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu_1, = 3$, $\mu_2 = 4$.
From (7), π_2 is unique optimal if $\lambda_s \in (13.51, \infty)$.
From (8)-(10), π_3 is unique optimal if $\lambda_s \in (0, 12.90)$

Example 2: $k = 10$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu_1 = 30$, $\mu_2 = 40$.
From (7), π_2 is unique optimal if $\lambda_s \in (114.94, \infty)$.
From (8)-(10), π_3 is unique optimal if $\lambda_s \in (0, 11.38)$.

5. SPECIAL CASE ($k = 1$) OF EXPONENTIAL REPAIR TIMES

For exponential repair times ($k = 1$) the preceding analysis remains valid for policy π_3 but not for policy π_2. If $k > 1$ then states $(k-1, k-1, 1)$ and $(k-1, 0, 1)$ are distinct and π_2 takes different actions in these states. If $k = 1$ then these states are identical and π_2 is not well defined. However, the solution can be obtained in a similar manner [12] and is:

Since it has been assumed that $\mu_1 \le \mu_2$, the policy that always repairs component 1 before component 2 is optimal if $\lambda_s \in [\alpha_2, \infty)$ where

$$\alpha_2 = [\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_1\lambda_2]/(\mu_2 - \mu_1).$$

Otherwise, the policy π_3 which never reassigns the repairman is optimal.

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