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ON LARGE DEVIATIONS PROPERTIES OF SEQUENTIAL ALLOCATION PROBLEMS

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ABSTRACT

Let $Y_{kt}, t = 1, 2, \ldots,\ k = 1, \ldots, r$ be r sequences of i.i.d. random variables and $\overline{X}_n = 1/n \sum_{k=1}^{n} Y_{A(k), t_{A(k)}}$ the sample mean of an n-size sample, given an adaptive allocation rule $\pi = \{A(t), t = 1, 2, \ldots\}$. We show that if $E[e^{\theta Y_{kt}}]$ is finite in a neighborhood of $\theta = 0$, then, for all adaptive rules π, $\overline{\lim}_{n\to\infty} 1/n \log P^{\pi}[\overline{X}_n \in C] \leq -\inf_{x\in C} I(x)$, for all closed sets $C \subset \Re$ and $\underline{\lim}_{n\to\infty} 1/n \log P^{\pi}[\overline{X}_n \in O] \geq -\inf_{x\in O} J(x)$, for all open sets $O \subset \Re$, where $I(x)$ and $J(x)$ are rate functions independent of π.

1. INTRODUCTION. Consider statistical populations $E_1, \ldots, E_r$. With each E_k is associated a sequence $\{Y_{kn}, n = 1, 2, \ldots\}$ of i.i.d. random variables representing outcomes or samples from population E_k. A sequential or adaptive allocation rule π is a sequence of probability measures on appropriately defined probability spaces (c.f., Dynkin & Yushkevich (1979)), which specify the probability of selecting a population at time t, given the previous history of selections and outcomes. Typically the selection is made in such a way to maximize some measure of performance, such as the expected sum of the outcomes, etc..

Let the random variables A_t, X_t denote the population from which a sample is taken at time t and the outcome of the t^{th} sampling, respectively. Let $S_n = \sum_{t=1}^{n} X_t$, $M_n^{\pi}(\theta) = E^{\pi}[e^{\theta S_n}]$, and $c_n^{\pi}(\theta) = 1/n \log M_n^{\pi}(\theta)$. In this framework $c_n^{\pi}(\theta)$ does not necessarily converge as $n \to \infty$ (see example 1 below).

The lack of convergence of c_n^π implies that the sufficient conditions for the large deviations property (cf. Ellis (1985)) do not hold. Nevertheless, in this paper we show that a large deviations type property holds for the sequence X_n. Specifically, we prove that there exist upper and lower bounds for the exponential decay of the large deviation probabilities, which are independent of the form of the allocation rule π. However, the upper and lower bounds do not coincide. This is the reason that we call this result a *large deviations type property*.

2. PROOF OF THE LARGE DEVIATIONS PROPERTY. Let F_k denote the common distribution of Y_{kn}, $\mu_k = \int x dF_k(x)$ the corresponding mean, $M_k(\theta) = \int e^{\theta y} dF_k(y)$ the moment generating function and $c_k(\theta) = \log M_k(\theta)$. Without loss of generality we assume that $\mu_k \geq 0$, $\forall k = 1, \dots, r$ (if not true, we can add a sufficiently large constant to all random variables Y_{kn} without changing the probabilistic properties of the problem). For notational convenience we also assume that $\mu_1 \leq \mu_2 \leq \dots \leq \mu_r$. Let Q_n^π denote the probability distribution of the history of selections and outcomes up to the first n samples, $H_n = (A_1, X_1 \dots, A_n, X_n)$ under allocation rule π and Q^π the distribution of the complete history $H = (A_1, X_1, \dots)$. Also let P_n^π denote the probability distribution of $\overline{X}_n$ under π, where $\overline{X}_n = S_n/n$. If π is stationary (i.e., at any step the selection probabilities are independent of past history and current time) then, $\{X_t, t = 1, 2, \dots\}$ is a sequence of i.i.d. random variables with distribution $F^\pi = \sum_{k=1}^r \pi(k) F_k$. Therefore, $\overline{X}_n \to \mu^\pi = \sum_{k=1}^r \pi(k)\mu_k$, almost surely, as $n \to \infty$. In addition, $c_n^\pi(\theta) = c(\theta) := \log\left(\sum_{k=1}^r \pi(k) M_k(\theta)\right)$, thus $c_n^\pi(\theta)$ converges as $n \to \infty$ for all θ, and a large deviations property holds for P_n^π, with rate function $I(z) = \sup_{\theta \in \Re}(\theta z - c(\theta))$ (cf. Ellis (1984)). Specifically, for all closed sets C and open sets O, it is true that

$$\overline{\lim_{n \to \infty}} \frac{1}{n} \log P_n^\pi(C) \leq -\inf_{x \in C} I(x), \tag{2.1}$$

$$\underline{\lim_{n \to \infty}} \frac{1}{n} \log P_n^\pi(O) \geq -\inf_{x \in O} I(x). \tag{2.2}$$

However for nonstationary allocation rules, $c_n^\pi(\theta)$ need not converge. In particular, consider the following example.

Example 1. Let $r = 2$ and consider an allocation rule π_0, which takes 2^0 samples from E_1, 2^1 from E_2, 2^2 from E_1 and so on.

At the end of the $(2k)^{\text{th}}$ group of samples π_0 has taken $\sum_{i=0}^k 2^{2i} = (4^{k+1} - 1)/3$ samples from E_1 and $\sum_{i=0}^{k-1} 2^{2i+1} = 2(4^k - 1)/3$ samples from E_2, for a total of $n_{1,k} = 2(4^k) - 1$. Similarly, at the end of the $(2k+1)^{\text{st}}$ group the number of samples from E_1 and E_2 is $(4^{k+1} - 1)/3$ and $2(4^{k+1} - 1)/3$ respectively, and the total $n_{2,k} = 4^{k+1} - 1$.

Since the populations are independent, we find that

$$\begin{aligned} c^{\pi_0}_{n_{1,k}}(\theta) &= \frac{1}{n_{1,k}}\left[\frac{4^{k+1}-1}{3}c_1(\theta)+\frac{2(4^k-1)}{3}c_2(\theta)\right] \\ &= \left[\frac{4^{k+1}-1}{3(2\,(4^k)-1)}c_1(\theta)+\frac{2(4^k-1)}{3(2\,(4^k)-1)}c_2(\theta)\right] \end{aligned} \tag{2.3}$$

$$\begin{aligned} c^{\pi_0}_{n_{2,k}}(\theta) &= \frac{1}{n_{2,k}}\left[\frac{4^{k+1}-1}{3}c_1(\theta)+\frac{2(4^{k+1}-1)}{3}c_2(\theta)\right] \\ &= \left[\frac{4^{k+1}-1}{3(4^{k+1}-1)}c_1(\theta)+\frac{2(4^{k+1}-1)}{3(4^{k+1}-1)}c_2(\theta)\right]. \end{aligned} \tag{2.4}$$

The two subsequences $c_{n_{1,k}}, c_{n_{2,k}}$ have in general different limits as $k \to \infty$ ($2/3\, c_1(\theta) + 1/3\, c_2(\theta)$ and $1/3\, c_1(\theta) + 2/3\, c_2(\theta)$, respectively), thus, $c_n^{\pi_0}$ does not converge.

In the remaining of this section we show that, in spite of Example 1, P_n^{π} satisfies a large deviations type property, with generally different rate functions for the upper and lower bound. Before we prove the main theorem we introduce some definitions and intermediate results.

Let $I_k(z) = \sup_\theta (\theta z - c_k(\theta))$ be the Legendre–Fenchel transform of $c_k(\theta)$. Lemma 2.1 below summarizes some well known properties of $c_k(\theta)$ and $I_k(z)$, It is a restatement in our notation of Lemma 2.2.5 of Dembo & Zeitouni (1993).

Lemma 2.1 **1.** *$c_k(\theta)$ is convex in θ and $I_k(z)$ is convex in z. Also $I_k(z) \geq 0$, with equality if and only if $z = \mu_k$.*

2. *If $c_k(\theta) < \infty$ for some $\theta > 0$, then $\mu_k < \infty$ and for all $z \geq \mu_k$*

$$I_k(z) = \sup_{\theta \geq 0} (\theta z - c_k(\theta)). \tag{2.5}$$

Similarly, if $c_k(\theta) < \infty$ for some $\theta < 0$, then $\mu_k > -\infty$ and for all $z \leq \mu_k$

$$I_k(z) = \sup_{\theta \leq 0} (\theta z - c_k(\theta)). \tag{2.6}$$

3. *$c_k(\theta)$ is differentiable for all θ such that $c_k(\theta) < \infty$. Moreover, if $\theta_k(z)$ is defined as the solution of $c_k'(\theta) = z$ in θ, then,*

$$z = \frac{\int x e^{\theta_k(z)x} dF_k(x)}{\int e^{\theta_k(z)x} dF_k(x)}, \tag{2.7}$$

$$I_k(z) = \theta_k(z)\, z \; - \; c_k(\theta_k(z)) \tag{2.8}$$

and $\theta_k(z) \geq 0$ (≤ 0) for $z \geq \mu_k$ $(\leq \mu_k)$, with equality if and only if $z = \mu_k$.

Define

$$\overline{c}(\theta) = \max_{k=1,\ldots,r} c_k(\theta) \tag{2.9}$$

$$\underline{c}(z) = \min_{k=1,\ldots,r} c_k(\theta_k(z)) \tag{2.10}$$

$$\overline{\theta}(z) = \max_{k=1,\ldots,r} \theta_k(z) \tag{2.11}$$

$$\underline{\theta}(z) = \min_{k=1,\ldots,r} \theta_k(z) \tag{2.12}$$

$$I(z) = \sup_{\theta} (\theta z - \overline{c}(\theta)) \tag{2.13}$$

$$J(z) = \begin{cases} \overline{\theta}(z)\ z - \underline{c}(z) & , \quad z \geq 0 \\ \underline{\theta}(z)\ z - \underline{c}(z) & , \quad z \leq 0 \end{cases} . \tag{2.14}$$

$I(z)$ and $J(z)$ are the rate functions for the upper and lower bound in Theorem 2.3. Lemma 2.2 states their properties.

Lemma 2.2 **1.** *$I(z) \geq 0$ for all z, with equality if and only if $\mu_1 \leq z \leq \mu_r$. Moreover $I(z)$ is nondecreasing for $z \geq \mu_r$ and nonincreasing for $z \leq \mu_1$.*

2.

$$I(z) \leq \min_k I_k(z) \leq \max_k I_k(z) \leq J(z). \tag{2.15}$$

Proof. Let $h_k(\theta, z) = \theta z - c_k(\theta)$ and $h(\theta, z) = \min_{k=1,\ldots,r} h_k(\theta, z)$. Then, $\forall z \in \Re$, $\forall k = 1, \ldots, r$, $I_k(z) = h_k(\theta_k(z), z)$ and $I(z) = \sup_{\theta\in\Re} h(\theta, z)$. By Lemma 2.1, $h_k(0, z) = 0$, and h_k is concave in θ for all k, z, thus, $h(\theta, z)$ is also concave in θ (minimum of concave functions).

To show part 1, consider three cases.

<u>Case 1.</u> $z > \mu_r$. Then, $\theta_k(z) > 0$ and $h_k(\theta_k(z), z) = I_k(z) > 0$, $k = 1, \ldots, r$, Thus, $h(\theta, z) > 0$, $\forall \theta \in (0, \underline{\theta}(z))$. In addition, $h(\theta, z) \leq 0$, $\forall \theta < 0$, because h is concave in θ and $h(\theta, z) = 0$. Therefore, $I(z) > 0$ and $I(z) = \sup_{\theta \geq 0} h(\theta, z)$. Also, $\forall z_1 \geq z_2 \geq \mu_r$, $\forall \theta > 0$, $h(\theta, z_1) \geq h(\theta, z_2)$, thus, $I(z_1) \geq I(z_2)$.

<u>Case 2.</u> $\mu_1 \leq z \leq \mu_r$. From Lemma 2.1, the maximizing point of $h_1(\theta, z)$ satisfies $\theta_1(z) \geq 0$, and since $h_1(\theta, z)$ is concave in θ, it follows that $h(\theta, z) \leq h_1(\theta, z) \leq 0$, $\forall \theta \leq 0$. Similarly, $\theta_r(z) \leq 0$ and $h(\theta, z) \leq h_r(\theta, z) \leq 0$, $\forall \theta \geq 0$. Thus, $h(\theta, z) \leq 0$, $\forall \theta$ and $h(0, z) = 0$, from which it follows that $I(z) = 0$.

<u>Case 3.</u> $z \leq \mu_1$. Following the same reasoning as in Case 1, it can be shown that $I(z) > 0$ and nonincreasing.

To show part 2, first note that $h(\theta, z) \leq h_k(\theta, z)$, $\forall k$, therefore, $I(z) = \sup_\theta h(\theta, z) \leq \sup_\theta h_k(\theta, z) = I_k(z)$, $\forall k$. In addition, $\forall z \geq 0$, $J(z) = \overline{\theta}(z) z - \underline{c}(z) \geq \theta_k(z) z - c_k(\theta_k(z) =$

$I_k(z)$, $\forall k$ and the same holds for $z \leq 0$, thus, (2.15) follows. □

Theorem 2.3 1. *For all closed sets* $C \subset \Re$

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log P_n^\pi C \leq - \inf_{x\in C} I(x). \tag{2.16}$$

2. *For all open sets* $O \subset \Re$

$$\underline{\lim}_{n\to\infty} \frac{1}{n} \log P_n^\pi O \geq - \inf_{x\in O} J(x). \tag{2.17}$$

Proof. Let $z > \mu_r$. We will show that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log P_n^\pi [z,\infty) \leq -I(z). \tag{2.18}$$

From the generalized Chebycheff inequality, $\forall \theta > 0$,

$$P_n^\pi[z,\infty) \;=\; Q_n^\pi[\overline{X}_n \geq z] \;=\; Q_n^\pi[\theta S_n \geq n\theta z] \;\leq\; e^{-n\theta z} E^\pi[e^{\theta S_n}]. \tag{2.19}$$

It is shown in Lemma 2.4 that

$$E^\pi[e^{\theta S_n}] \leq e^{n\overline{c}(\theta)}. \tag{2.20}$$

Combining (2.19) and (2.20),

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log P_n^\pi[z,\infty) \leq -\sup_{\theta>0}[\theta z - \overline{c}(\theta)] = -I(z) = -\inf_{y\in[z,\infty)} I(y).$$

The extension to any closed set C is shown following an argument in Varadhan (1984). If $C \bigcap [\mu_1, \mu_r] \neq \emptyset$, then, from Lemma 2.2, $\inf_{x\in C} I(x) = 0$. If $C \cup [\mu_1,\mu_r] = \emptyset$, let $y_1 = \max\{y : y \in C, y < \mu_1\}$ and $y_2 = \min\{y : y \in C, y \geq \underline{\mu}\}$. Since C is closed, y_1, y_2 always exist (with the convention $\max\emptyset = -\infty$, $\min\emptyset = \infty$). Then, $P_n^\pi C \leq 2\max\{P_n^\pi(-\infty, y_1], P_n^\pi[y_2,\infty)\}$, therefore, $\overline{\lim}_{n\to\infty} \frac{1}{n}\log P_n^\pi C \leq -\min\{I(y_1), I(y_2)\} = -\inf_{x\in C} I(y)$, which proves (2.16).

We next show (2.17). For any open set O and $z \in O$, $\exists\, \delta > 0$ sufficiently small, such that $B_{z,\delta} \subset O$, where $B_{z,\delta} = \{x \in \Re : |x - z| < \delta\}$. Therefore, $P_n^\pi O \geq P_n^\pi B_{z,\delta}$, and, in order to show (2.17), it suffices to show that, $\forall z$,

$$\underline{\lim}_{n\to\infty} \frac{1}{n} \log P_n^\pi B_{z,\delta} \geq -J(z). \tag{2.21}$$

Fix $z > \mu_r$ and δ sufficiently small. To show (2.21) it suffices to show that

$$\underline{\lim}_{n\to\infty} \frac{1}{n} \log Q_n^\pi B'_{z,\delta} \geq -J(z), \tag{2.22}$$

where

$$B'_{z,\delta} = \{h_n \ : \ \overline{X}_n \in B_{z,\delta},\ S_{k,t_k(n)} \geq 0, k = 1, \ldots, r\}, \tag{2.23}$$

$S_{k,t_k(n)} = \sum_{t=1} t_k(n) y_{k,t}$ and $t_k(n) = \sum_{t=1}^{n} \mathbf{1}\{A_t = k\}$. Quantities $t_k(n)$ and $S_{k,t_k(n)}$ denote the number and the sum of the observations from population E_k during the first n periods. Recall that Q_n^π represents the probability distribution of the n − −period history H_n.

For $k = 1, \ldots, r$, define a measure transformation $\tilde{F}_{k,z}$ of F_k as

$$d\tilde{F}_{k,z}(x) = \frac{e^{\theta_k(z)x} dF_k(x)}{M_k(\theta_k(z))}. \tag{2.24}$$

Then, by definition of $\theta_k(z)$,

$$\int x d\tilde{F}_{k,z}(x) = z, k = 1, \ldots, r. \tag{2.25}$$

Conditioning on history h_n and using the transformation (2.24),

$$\begin{aligned} Q_n^\pi B'_{z,\delta} &= \int_{B'_{z,\delta}} dQ_n^\pi(h_n) \\ &= \int_{B'_{z,\delta}} \left(\prod_{t=1}^{n} M_{a_t}(\theta_{a_t}(z)) e^{-\theta_{a_t}(z)} \right) d\tilde{Q}_n^\pi(h_n) \\ &= \int_{B'_{z,\delta}} \exp\left(\sum_{k=1}^{r} t_k(n) c_k(\theta_k(z)) - \theta_k(z) S_{k,t_k(n)} \right) d\tilde{Q}_n^\pi(h_n), \end{aligned} \tag{2.26}$$

where $\tilde{Q}_n^\pi$ denotes the distribution of H_n when the populations follow the modified distributions $\tilde{F}_{k,z}$.

In Lemma 2.5 it is shown that, first, on $B'_{z,\delta}$,

$$\sum_{k=1}^{r} \theta_k(z) S_{k,t_k(n)} \leq n\overline{\theta}(z)\ (z + \delta). \tag{2.27}$$

and second,

$$\lim_{n \to \infty} \frac{1}{n} \log \tilde{Q}_n^\pi B'_{z,\delta} = 0. \tag{2.28}$$

Using (2.27),(2.28) and the definition of $\underline{c}(z)$, (2.26) becomes

$$Q_n^\pi B'_{z,\delta} \ \geq \ e^{n\underline{c}(z)} e^{-\overline{\theta}(z)(nz+n\delta)} \tilde{Q}_n^\pi B_{z,\delta} \ = \ e^{-nJ(z)-n\delta} \tilde{Q}_n^\pi B'_{z,\delta}, \tag{2.29}$$

thus, $1/n \log Q_n^\pi B'_{z,\delta} \geq -J(z) - \delta + 1/n \log \tilde{Q}_n^\pi B'_{z,\delta}$, from which (2.22) follows by letting $n \to \infty$ and $\delta \to 0$.

In order to complete the proof of (2.21), we also need to consider the cases $0 \leq z \leq \mu_1$, $\mu_1 \leq z < \mu_r$ and $z < 0$. These can be handled in the same way as the case $z > \mu_r$, with the appropriate modifications of the definition of $B'_{z,\delta}$ and relation (2.27). □

Lemma 2.4 *Under any adaptive allocation rule* π, $E^\pi[e^{\theta S_n}] \le e^{n\bar{c}(\theta)}$.

Proof. Conditioning on the $(n-1)$– –history $H_{n-1} = h_{n-1} = a_1, x_1, \ldots, a_{n-1}, x_{n-1}$,

$$\begin{aligned} E^\pi[e^{\theta S_n}] &= E^\pi[E^\pi[e^{\theta S_{n-1}} e^{\theta X_n} | H_{n-1} = h_{n-1}]] \\ &= E^\pi[e^{\theta S_{n-1}} E^\pi[e^{\theta X_n} | H_{n-1} = h_{n-1}]]. \end{aligned} \tag{2.30}$$

Given policy π and history h_{n-1}, the distribution of X_n is F_k, with probability $\pi_n(k|h_{n-1})$, for $k = 1, 2, \ldots, r$. Therefore

$$\begin{aligned} E^\pi[e^{\theta X_n} | H_{n-1} = h_{n-1}] &= \sum_{k=1}^{r} \pi_n(k|h_{n-1}) E[e^{\theta Y_{k1}}] \\ &= \sum_{k=1}^{r} \pi_n(k|h_{n-1}) M_k(\theta) \\ &\le \max_k M_k(\theta) = e^{\bar{c}(\theta)}. \end{aligned} \tag{2.31}$$

Applying (2.31) repeatedly to (2.30), the lemma follows. □

Lemma 2.5 *Let* $z > \mu_r$ *and define the set* $B'_{z,\delta}$ *as in (2.23). Then*

1. *On the event* $B'_{z,\delta}$ *it isa true that* $\sum_{k=1}^r \theta_k(z) S_{k,t_k(n)} \le n\bar{\theta}(z)\ (z+\delta)$.
2. *Let* $\tilde{Q}_n^\pi$ *denote the distribution of* H_n *when the populations follow the modified distributions* $\tilde{F}_{k,z}$. *Then,* $\lim_{n\to\infty} \frac{1}{n} \log \tilde{Q}_n^\pi B'_{z,\delta} = 0$.

Proof. Since $z > \mu_r \ge 0$, it follows from Lemma 2.2 that $\theta_k(z) > 0$, $\forall k$. In addition, on the event $B'_{z,\delta}$ it is true that

$$\begin{aligned} &nz - n\delta < \sum_{k=1}^{r} S_{k,t_k(n)} < nz + n\delta, \\ &S_{k,t_k(n)} \ge 0, \quad \forall k. \end{aligned}$$

Let

$$N = \max\{\sum_{k=1}^{r} \theta_k(z) S_k \ :\ \sum_{k=1}^{r} S_k \le nz + n\delta,\ \sum_{k=1}^{r} S_k \ge nz - nd, S_k \ge 0\}.$$

The solution of the above optimization problem corresponds to an extreme point of the (convex) feasible region. Any extreme point is of the form $S_l = nz - n\delta$, $S_j = 0, \forall j \ne l$, or $S_l = nz + n\delta$, $S_j = 0, \forall j \ne l$, for some $l = 1, \ldots, r$. Therefore,

$$\begin{aligned} N &= \max_{k=1,\ldots,r} \{ \max\{\theta_k(z)(nz - n\delta),\ \theta_k(z)(nz + n\delta)\} \} \\ &= \max_{k=1,\ldots,r} \{\theta_k(z)(nz + n\delta)\} \\ &= \bar{\theta}(z)(nz + n\delta), \end{aligned}$$

from which part 1 follows.

We next show part 2. Note that

$$\tilde{Q}_n^\pi B'_{z,\delta} = \tilde{Q}_n^\pi[\overline{X}_n \in B_{z,\delta} \mid S_{k,t_k(n)} \geq 0,\ \forall k]\ \tilde{Q}_n^\pi[S_{k,t_k(n)} \geq 0,\ \forall k]\ ,$$

thus,

$$\frac{1}{n}\log \tilde{Q}_n^\pi B'_{z,\delta} = \frac{1}{n}\log \tilde{Q}_n^\pi[\overline{X}_n \in B_{z,\delta} \mid S_{k,t_k(n)} \geq 0,\ \forall k] + \frac{1}{n}\log \tilde{Q}_n^\pi[S_{k,t_k(n)} \geq 0,\ \forall k]. \quad (2.32)$$

Since the transformed distributions all have expectation equal to z, it follows from the law of large numbers that $\lim_{n\to\infty} \tilde{Q}_n^\pi[\overline{X}_n \in B_{z,\delta}] = 1$. Therefore, $\lim_{n\to\infty} \tilde{Q}_n^\pi[\overline{X}_n \in B_{z,\delta} | S_{k,t_k(n)} \geq 0,\ \ \forall k] = 1$, and

$$\lim_{n\to\infty} \frac{1}{n}\log \tilde{Q}_n^\pi[\overline{X}_n \in B_{z,\delta} \mid S_{k,t_k(n)} \geq 0,\ \forall k] = 0. \quad (2.33)$$

Next conider $\tilde{Q}_n^\pi[S_{k,t_k(n)} \geq 0,\ \forall k]$. The event $\{S_{k,t} \geq 0,\ \forall t, \forall k\}$ implies the event $\{S_{k,t_k(n)} \geq 0,\ \ \forall k\}$, therefore

$$\tilde{Q}_n^\pi[S_{k,t_k(n)} \geq 0,\ \ \forall k] \geq \tilde{Q}^\pi[S_{k,t} \geq 0,\ \ \forall t = 1,2,\ldots, \forall k = 1,\ldots,r].$$

Consider the "round robin" allocation rule which starts with one sample from E_1 and then takes one sample from each population repeatedly. As $n \to \infty$ this rule will observe all infinite sequences of outcomes from each population. Therefore, if under this rule the event $\{S_{k,t} \geq 0,\ \ \forall k\}$ is realized, then it will also be realized under any adaptive rule π. In addition, under the round robin rule, $S_{k,t}, t = 1,2,\ldots,\ k = 1,\ldots,r$ represent the partial sums of r independent sequences of i.i.d random variables, each with distribution $\tilde{F}_{k,z}$. From these observations it follows that

$$\tilde{Q}^\pi[S_{k,t} \geq 0,\ \ \forall t = 1,2,\ldots,\ k = 1,\ldots,r] \geq \prod_{k=1}^{r} P^{\tilde{F}_k}[S_{k,t} \geq 0,\ \ \forall t = 1,2,\ldots].$$

Under the transformed distribution $\tilde{F}_{k,z}$, $Y_{k,t}, t = 1,2,\ldots$ is a sequence of i.i.d. random variables with mean $z > 0$. From Lemma 2.8 of Chow, Robbins & Siegmund (1971) we obtain $P^{\tilde{F}_k}[S_{k,t} \geq 0,\ \ \forall t = 1,2,\ldots] > 0,\ \ \forall k$, thus, $\tilde{Q}_n^\pi B'_{z,\delta} \geq Q := \prod_{k=1}^r P^{\tilde{F}_k}[S_{k,t} \geq 0,\ \ \forall t = 1,2,\ldots] > 0$, and

$$\varliminf_{n\to\infty} \frac{1}{n}\log \tilde{Q}_n^\pi B'_{z,\delta} \geq \frac{1}{n}\log Q = 0.$$

On the other hand, since $\tilde{Q}_n^\pi B'_{z,\delta} \leq 1$, $\varlimsup_{n\to\infty} \frac{1}{n}\log \tilde{Q}_n^\pi B'_{z,\delta} \leq 0$. Therefore,

$$\lim_{n\to\infty} \frac{1}{n}\log \tilde{Q}_n^\pi B'_{z,\delta} = 0. \quad (2.34)$$

Part 2 follows from (2.32), (2.33) and (2.34). □

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