STOCHASTIC ANALYSIS AND APPLICATIONS, 13(2), 137-164 (1995)

ON THE OPTIMAL MAINTENANCE OF SYSTEMS AND CONTROL OF ARRIVALS IN QUEUES

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ABSTRACT

Consider a continuously operating system. A problem that arises in practice and is related to the optimal operation of the system is how to allocate limited repair resources in a most efficient manner, i.e., to maximize a measure of system performance. In this article our interest is in identifying classes of reliability systems (structures) such that optimal dynamic repair allocation policies depend on the structure only. A general structure of a system is that of a K-*out-of*-N system where component i is itself assumed to be a K_i'-*out-of*-N_i' system. All subsystems have identical components the lifetimes of which are assumed to be independent, exponentially distributed random variables. The system is maintained by a single repairman and the time it takes to repair a failed component is also an exponentially distributed random variable. Repaired components are as good as new and preemptions are allowed. Using several optimality criteria, we characterize all stationary optimal policies for the following structures: the K-*out-of*-N system with 1-*out-of*-N_i' subsystems, the K-*out-of*-N system with N'-*out-of*-N' subsystems, the N-*out-of*-N system with K'-*out-of*-N_i' subsystems, and the 1-*out-of*-N system with K'-*out-of*-N' subsystems. We discuss the implications of the results obtained to problems of control of arrivals in queues.

1. INTRODUCTION.

Consider a system consisting of N subsystems which functions when at least K out of its N component-subsystems are operational. Subsystem i consists of N_i' components and functions when at least K_i' of them operate. This structure will be denoted by $[K \mid N ; (K_i' \mid N_i')_{i=1,\ldots,N}]$. Components of the subsystems may fail. Their lifetimes are exponentially distributed, independent random variables.The rate of failure is common for all components of all subsystems and is denoted by μ. Failures may occur even when the system is not functioning. The system is maintained by a single repairman who may be assigned to any failed component (from now on component will always mean "component of a subsystem"). The repairman may switch from one failed component to another instantaneously and the time it takes him to complete the repair of any failed component is considered to be an exponentially distributed random variable with parameter λ. Repaired components are as good as new. Extreme cases of such structures are (a) a series structure of subsystems, where each subsystem is a parallel system and (b) a parallel structure of subsystems, where each subsystem is a series system. For (a) it is intuitively expected that a good repair allocation policy should always assign the repairman to any subsystem among those with the smaller number of functioning components, i.e., it will tend to ***equalize*** the number of functioning components across subsystems; where if some subsystems have an equal number of functioning components the policy breaks ties arbitrarily. We call it ***equalizing policy***. For (b) a good policy should tend to maximize the number of series subsystems that are functioning, where ties are also broken arbitrarily. This policy, which is clearly the opposite to the equalizing policy, tends to maximize the number of functioning components of the subsystem that is "more" operational. We call it ***inequalizing policy***. It turns out that the inequalizing policy is optimal without any additional restrictions only when the subsystems are identical. Hence we see that it is important to be able to identify precisely the most general structures for which optimal policies do posses these type of properties. This is done for the $[K \mid N ; (1 \mid N_i')_{i=1,\ldots,N}]$ system (case (1.A)), the $[N \mid N ; (K' \mid N_i')_{i=1,\ldots,N}]$ system (case (1.B)), the $[1 \mid N ; (K' \mid N')_{i=1,\ldots,N}]$ system (case (2.A)) and the $[K \mid N ; (N' \mid N')_{i=1,\ldots,N}]$, system (case (2.B)). Note that in cases (1.A) and (1.B) , subsystems are not assumed to be identical, but the necessary amount of functioning components for a subsystem to function is assumed to be the same for all subsystems. For cases (2.A) , (2.B) , subsystems are assumed to be identical. We establish the optimality of a pertinent policy for each class under various criteria and show they are unique optimal among all policies. Specifically we show the following.

For the $[K \mid N; (1 \mid N_i')_{i=1,\ldots,N}]$ system, the equalizing policy possesses the following optimality properties. It maximizes *stochastically the number of working subsystems at any time instant t*. Hence, it maximizes both the *expected discounted system operation time for all discount rates* β $(\beta > 0)$ and the *average system operation time*.

For the $[N \mid N; (K' \mid N_i')_{i=1,\ldots,N}]$ system, consider the generalization of the equalizing policy defined as follows. Whenever the system is down it assigns the repairman to any non-functioning subsystem and when the system is up it follows the equalizing policy. We call this policy K' – *equalizing* policy and show that it maximizes the *probability the system is up at any time instant t* (i.e., the *Reliability* of the system). It follows that it also maximizes both the *expected discounted system operation time for all discount rates* β $(\beta > 0)$ and the *average system operation time*.

For the $[1 \mid N; (K' \mid N')_{i=1,\ldots,N}]$ system the inequalizing policy possesses the following optimality properties. It maximizes the *probability the system is up at any time instant t*, hence, it also maximizes both the *expected discounted system operation time for all discount rates* β $(\beta > 0)$ and the *average system operation time*.

For the $[K \mid N; (N' \mid N')_{i=1,\ldots,N}]$ system, consider the generalization of the inequalizing policy defined as follows. i) Whenever the system is down and there are less than K subsystems with functioning components it assigns the repairman to any non functioning subsystem (until the system reaches a state in which there are exactly K subsystems with functioning components). ii) Whenever the system is down and there are exactly K subsystems with functioning components it assigns the repairman to any non functioning subsystem with functioning components. iii) Otherwise, it follows the inequalizing policy. We call this policy K-*inequalizing* policy and show that it maximizes the *probability that the subsystem is up at any time instant t*. Hence, it also maximizes both the *expected discounted system operation time* for all discount rates β $(\beta > 0)$ and the *average system operation time*.

Furthermore, it is shown that when an assignment choice is made between two subsystems with equal number of working components, it is immaterial to which subsystem the repairman is assigned for all classes. Up to this indifference, the pertinent policies in the above classes of problems, are shown to be *uniquely* optimal, i.e., any policy that does not have the pertinent structure is strictly suboptimal for the corresponding class.

The above repair allocation models posses the following ("dual") interpretation in terms of queuing theory and scheduling. Consider a queuing system with N servers (service stations). Server i may work on up to N'_i customers simultaneously, $N'_i \leq \infty$; we call N'_i the *capacity* of server i. Customers, whose service times are independent identically distributed exponential random variables with parameter μ, arrive according to a Poisson process with parameter λ. Arriving customers that find all servers busy are lost. Upon arrival customers are assigned to a server (if there is one with idle capacity) and they may not change server thereafter. The problem is to assign the arriving customers to servers according to a policy that is optimal with respect to certain performance criteria. In this context, *equalizing* means to assign the current arrival to the server with maximum idle capacity available currently, in*equalizing* means to assign the current arrival to the server with minimum idle capacity available currently, *system down* means that *there is at least one server with less than* K' *customers being served* (case (1.b)) or that *there are less than* K servers working at full capacity (case (2.b). We establish the following results. (1.a) The equalizing policy *minimizes stochastically the number of idle servers at any time instant t.* (1.b) The K – equalizing policy *minimizes the probability there is at least one server with less than* K, $0 < K < N'_i$, customers at any time instant t. (2.a) The inequalizing policy *minimizes the probability that all servers have less than* K, $0 < K < N'$, *customers at any time instant t.* In this case we have to assume that all servers have identical capacity N'. (2.b) The K-inequalizing policy minimizes the *probability that there are less than* K servers working at full capacity at any time instant t. In this case we also have to assume that all servers have identical capacity N'. Furthermore, it is shown that when an assignment choice is made between two servers with equal number of customers being served, it is immaterial to which subsystem the repairman is assigned. Up to this indifference, the pertinent policies in the above classes of problems, are shown to be *uniquely* optimal.

For related work in this area we refer to Smith (1978 a,b), Derman, Lieberman and Ross (1978), Nash and Weber(1982), Katehakis and Derman (1984, 1989), Courcoubetis and Varaiya (1984), Katehakis and Melolidakis (1988, 1989), Frosting (1992), Hordijk A. and G. Koole (1992) and references given there.

2. PROBLEM FORMULATION.

Consider the reliability model first. The assumptions made imply that at any time instant, the status of all subsystems is given by a vector $x=(x_1,x_2,\ldots,x_N)$ with x_i denoting the number of functioning components of subsystem i, $0\leq x_i\leq N_i$ for $i=1,\ldots,N$. The set of all such vectors is the set of states of the system and is denoted by S. The structure of the system is specified by a partition of the state space S into two sets G and B of "good" and "bad" states or alternatively, by the structure function $\phi(x)$ (c.f. Barlow and Proshan (1977)). For $x\in S$, let $C_i^0(x)$ ($C_i^1(x)$) denote the set of failed (functioning) components of subsystem i. We define $C^0(x):=\bigcup_{i=1}^N C_i^0(x)$ (the set of all failed components). Let also $A(x):=\{i: C_i^0(x)\neq\emptyset\}$ denote the set of subsystems with failed components at state x. The cardinality of set A is denoted by $|A|$ and the number of working subsystems at state x is denoted by $M_\phi(x)$. We define the state (δ_i,x) by

$$(\delta_i,x)=\begin{cases}(x_1,\ldots,x_{i-1},x_i+\delta,x_{i+1},\ldots,x_N) & \text{if } \delta\in\{-x_i,-x_i+1,\ldots,N_i'-x_i\}\\ x & \text{otherwise}\end{cases}\tag{1}$$

To simplify the notation, whenever ambiguities may not arise, we will not repeat x in notations like $|C_{a(x)}^0(x)|$, i.e. we will write $|C_a^0(x)|$ instead of $|C_{a(x)}^0(x)|$.

To describe the optimal policies we need some more notation. Define the correspondences $a(x)$, $a_{K'}(x)$ $b(x)$, $b_K(x)$ of S into $2^{\{1,\ldots,N\}}$ such that when the system is in state x, $a(x)$ is any subsystem with the least amount of functioning components for the $[K\,|\,N;(1\,|\,N_i')_{i=1,\ldots,N}]$ system, $a_{K'}(x)$ is any non-functioning subsystem if the system is down; otherwise, $a_{K'}(x)$ is any subsystem with the least amount of functioning components for the $[N\,|\,N;(K'\,|\,N_i')_{i=1,\ldots,N}]$ system, $b(x)$ is any subsystem with the largest amount of functioning components at state x, such that $C_b^0(x)\neq\emptyset$ for the $[1\,|\,N;(K'\,|\,N')_{i=1,\ldots,N}]$ system. Finally, $b_K(x)$ is associated with the $[K\,|\,N;(N'\,|\,N')_{i=1,\ldots,N}]$ system and it is defined to be any non-functioning subsystem, if the system is down at state x and there are less than K subsystems that have at least one operating component. If the system is down at state x and there are exactly K subsystems that have at least one operating component, $b_K(x)$ is any nonfunctioning subsystem among those that have at least one operating component; otherwise, $b_K(x)$ is any subsystem with the largest amount of functioning components.

Note that

$$x_{a(x)}=\min\{x_j,\ j=1,\dots,N\}, \tag{2}$$

$$a_{K'}(x)=\begin{cases} \text{any } j,\ j=1,\dots,N, \text{ such that } x_j < K' & \text{if } M_\phi(x) < N \\ a(x) & \text{otherwise.} \end{cases} \tag{3}$$

$$x_{b(x)}=\max\{x_j,\ x_j \neq\ j=1,\dots,N\} \tag{4}$$

$$b_K(x)=\begin{cases} \text{any } j,\ j=1,\dots,N, \text{ such that } x_j < N', & \text{if } M_\phi(x) < K \text{ and } |P(x)| < K \\ \text{any } j \in P(x) \text{ such that } x_j < N', & \text{if } M_\phi(x) < K \text{ and } |P(x)| = K \\ b(x) & \text{otherwise.} \end{cases} \tag{5}$$

where in the last definition we use the notation $P(x)=\{j: x_j > 0\}$. Note also that $a_1(x) \equiv a(x)$ and $b_1(x) \equiv b(x)$.

Let ϖ be a permutation of the set of subsystems, $\{1,2,\dots,N\}$. Then, a *permissible* permutation ϖ of a state x is defined by $\varpi(x):=(x_{\varpi^{-1}(1)}, x_{\varpi^{-1}(2)},\dots, x_{\varpi^{-1}(N)})$, where $x_{\varpi^{-1}(i)} \leq N_i$ $i=1,2,\dots,N$. For example, if the state of the system is $(2,3,0)$ and subsystem 1 consists of 2 components, subsystem 2 of 4 components, and subsystem 3 of 3 components, then $(0,2,3)$ is a permissible permutation, while $(3,2,0)$ is not. Since the subsystems have identical components a deterministic policy (c.f. Derman (1970)) need to be specified only up to the subsystem on which a repairman is assigned to. Given a policy π, we define the stochastic process: $x_\pi(t)$, $N_\pi(t;x)$, $Z_\pi(t;x)$ as follows.

i) $x_\pi(t)=(x_{1\pi}(t),\dots,x_{N\pi}(t))$ takes values in S and represents the status of all subsystems at time t. *ii)* $N_\pi(t;x)= M_\phi(x_\pi(t))$ takes values in $\{0,1,\dots,N\}$ and represents the number of functioning subsystems at time t, provided the state at time 0 was x. *iii)* $Z_\pi(t;x)=\phi(x_\pi(t))$ takes values in $\{0,1\}$ and represents the status of the system at time t, provided the state at time 0 was x. Note that $\{x_\pi(t)\ ,\ t \geq 0\}$, $\{N_\pi(t;x)\ ,\ t \geq 0\}$, $\{Z_\pi(t;x)\ ,\ t \geq 0\}$ are all continuous time Markov chains if π is a

Markov policy (c.f. Derman (1970)). If there exist a policy π_0 such that $N_{\pi_0}(t;x) \overset{st}{\geq} N_\pi(t;x)$ (respectively, $Z_{\pi_0}(t;x) \overset{st}{\geq} Z_\pi(t;x)$) $\forall$ $t \geq 0$, for all policies π, then, we say that π_0 maximizes stochastically the number of functioning subsystems (respectively, the reliability of the system) at any time t given the initial state x. To establish the stochastic optimality of a policy π_0 with respect to $N_\pi(t;x)$ (respectively, $Z_\pi(t;x)$) one must show that the following inequalities (6) (respectively, (7)) hold.

$$P(N_{\pi_0}(t;x) > k) \geq P(N_\pi(t;x) > k), \quad \forall\, t \geq 0, \quad \forall k=0,1,\ldots,N, \quad \text{for all } \pi. \quad (6)$$

$$P(Z_{\pi_0}(t;x) = 1) \geq P(Z_\pi(t;x) = 1)), \quad \forall\, t \geq 0, \text{ for all policies } \pi. \quad (7)$$

The key idea in establishing (6), (7) above, is to observe that the random variables $N_\pi(t;x)$ and $Z_\pi(t;x)$ may change value at transition epochs only. Furthermore, we use the device of uniformization to equalize the times between transition epochs regardless of the policy being employed; see also Katehakis and Melolidakis (1988, 1989). Thus, at any state we consider (dummy) transitions back to the same state at such a rate that the sojourn times $X_1, X_2, \ldots$ of the process in the different states are identically distributed random variables with rate r, where r is an upper bound on the transition rates of the process. This resulting process is probabilistically identical with the original one (c.f. Lippman (1975), Ross (1983)). Note that, even though the number of transition epochs is enlarged by the uniformization, no complication arises as far as policies are concerned when one can restrict attention to deterministic policies. Since in our models the state space is finite, the uniformized process exists. In the sequel we will restrict attention to the latter and in addition, by an appropriate change of the unit of time, we will assume that $r=1$. Let $S_n = X_1 + \cdots + X_n$ denote the (random) time of the n^{th} transition epoch ($S_0=0$), and let $n(t)=\sup\{n : S_n \leq t\}$. Since we are considering the uniformized process, S_n and $n(t)$ are independent of the policy being used and it is easy to see that $\{n(t),\ t \geq 0\}$ is a Poisson process with rate 1. Let $\tilde{N}_\pi(n;x)$ (respectively, $\tilde{Z}_\pi(n;x)$) denote the number of functioning subsystems (respectively, the status of the system) at the start of the n^{th} transition epoch. Note that $N_\pi(t;x)=\tilde{N}_\pi(n(t);x)$ and $Z_\pi(t;x)=\tilde{Z}_\pi(n(t);x)$ $\forall t \geq 0$. Furthermore,

$$P(\tilde{N}_\pi(n(t);x) \geq k) = \sum_{n=0}^{\infty} P(\tilde{N}_\pi(n;x) \geq k) P(n(t)=n), \text{ for all } \pi,\ 0 \leq k \leq N. \quad (8)$$

Let us define as $(\Pi_{n,k})$ the problem of maximizing $P(\tilde{N}_\pi(n;x) \geq k)$, with respect to π for fixed n and k ; $n=0,1,\ldots$, $k=0,1,\ldots,N$. It follows that if there exists a deterministic policy π_0 (whose actions are independent of n and k) that is optimal for all problems $(\Pi_{n,k})$ then this policy maximizes stochastically $N_\pi(t;x)$. Furthermore if the problems $(\Pi_{n,k})$ posses a unique optimal policy then this is the only policy that maximizes stochastically $N_\pi(t;x)$. Analogous arguments hold for the problem of maximizing stochastically $Z_\pi(t;x)$, i.e., it suffices to consider the problems $(\Pi'_{n,k})$, of maximizing $P(\tilde{Z}_\pi(n;x) = 1)$; $n=0,1,\ldots$. For each fixed n and k the problem $(\Pi_{n,k})$ (respectively the problem $(\Pi'_{n,k})$) is specified by the following elements. 1) *State space:* $\{(x;m): x\in S,\ m=0,1,\ldots,n\}$. 2) *Action sets:* $A(x;m)=A(x)$. 3) *System dynamics:* when the system is in state $(x;m)$ and action i is chosen, then, the possible transitions are: i) to state $(1_i,x;m-1)$ with probability λ, ii) to state $(-1_l,x;m-1)$, for all subsystems l such that $C_l^1(x)\neq\emptyset$, with probability μx_l , *iii) to state* $(x;m-1)$, with probability $(1-\lambda-\mu(x))$, where $\mu(x)=\mu\sum_{l=1}^{N} x_l$. 4) *Reward structure* $r(x,m)$ (respectively $r'(x,m)$) given by,

$$r(x,m)=\begin{cases}1 & \text{if } M_\phi(x)\geq k \text{ and } m=0\\ 0 & \text{otherwise}\end{cases}, \tag{9}$$

and

$$r'(x,m)=\begin{cases}1 & \text{if } \phi(x)=1 \text{ and } m=0\\ 0 & \text{if } \phi(x)=0\end{cases} \tag{10}$$

Now, let $H_1(x)=r(x,0)$ and $H_2(x)=r'(x,0)$. Then, the dynamic programming equations for $(\Pi_{n,k})$ (respectively $(\Pi'_{n,k})$) , are the following

$$v(x;m+1)=\max_{i\in A(x)}\Big([1-\lambda-\mu(x)]\,v(x;m)+\lambda\, v(1_i,x;m)+\mu\sum_{l=1}^{N} x_l\, v(-1_l,x;m)\Big) \qquad (11)_{m+1}$$

$m=0,\ldots,n-1$.

$$v(x;0)=H_j(x)\ , \qquad (11)_0$$

for $j=1$ (respectively for $j=2$) .

Let $w_\pi(x,m)$ denote the value function of a deterministic policy for $(\Pi_{n,k})$ (respectively $(\Pi'_{n,k})$) ; it is the unique solution to the following system of linear equations

$$w_\pi(x;m+1)=[1-\lambda-\mu(x)]w_\pi(x;m)+\lambda\, w_\pi(1_{\pi(x,m+1)},x;m)+\mu\sum_{l=1}^{N}x_l\, w_\pi(-1_l,x;m)$$

$$m=0,\dots,n-1\ , \qquad (12)_{m+1}$$

$$w_\pi(x;0)=H_j(x)\ , \qquad (12)_0$$

for $j=1$ (respectively for $j=2$). Note that the policies under consideration (e.g., equalizing etc.) have the specification that $\pi(x,m)=\pi(x)$, for all m.

We can now state the following

PROPOSITION 2.1. A necessary and sufficient condition for a stationary policy π_0 to be optimal for problem $(\Pi_{n,k})$ is the following

$$w_{\pi_0}(1_{\pi_0(x)},x;m)\geq w_{\pi_0}(1_i,x;m)\ ,\ \ \forall x\in S\ ,\ \ \forall i\in A(x)-\{\pi_0(x)\}\ \ ,\ \ m=0,1,\dots,n-1 \qquad (13)_m$$

PROOF. Equations $(13)_m$ imply that $w_{\pi_0}(x;m)$, $x\in S$, $m=1,\dots,n$, constitute a solution to the optimality equations $(11)_m$, $m=1,\dots,n$, which have a unique solution.

PROPOSITION 2.2. A necessary and sufficient condition for a policy π_0 to be the unique stationary optimal policy for the problem of maximizing stochastically $N_\pi(t;x)$ (respectively $Z_\pi(t;x)$) is the following

$\forall x\in S$, $\exists m_0=m_0(x)$ such that $(13)_{m_0}$ *is a strict inequality* $\forall i\in A(x)-\{\pi_0(x,m)\}$. (14)

PROOF. The claim follows from (8) and arguments analogous to those that lead to the definition of problems $(\Pi_{n,k})$, $(\Pi'_{n,k})$.

REMARK 2.1. The system performance criteria we use lead to optimal policies with rather strong optimality properties. In particular, all optimal policies for all cases of this paper will also be optimal with respect to the *total expected discounted operation time of the system* and with respect to the *availability of the system*. To see this notice (i) that for the structures under consideration, if a policy maximizes stochastically $N_\pi(t;x)$, then it also maximizes stochastically $Z_\pi(t;x)$, and (ii) The total expected discounted operation time of the system is given by $\int_0^\infty e^{-\beta t}P(Z_\pi(t;x)=1)\,dt$ and the availability of the system is given by $\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-\beta t}P(Z_\pi(t;x)=1)\,dt$.

3. REPAIR ALLOCATION MODELS.

In this section we establish the optimality of the pertinent policies for the classes of systems introduced in Part 2. We start with the $[K \mid N; (1 \mid N_i')_{i=1,\ldots,N}]$ system. First note that

$$N_\pi(t;x) = M_\phi(x_\pi(t)) \tag{15}$$

where $M_\phi(x_\pi(t)) = |\{i : x_{i\pi}(t) \geq 1\}|$. We can now state the following.

PROPOSITION 3.1. (a) *The equalizing policy maximizes stochastically $N_\pi(t;x)$, at any time instant t.* (b) *Any policy that is not equalizing is strictly worse than the equalizing policy.*

PROOF. The proof is by induction on m using (2.13) and (2.12) in Lemmata 3.1.1, 3.1.2 below. The proof of (b) is given in Lemma 3.1.3 below.

For notational simplicity we will use the notation $w(x,m)$ in place of $w_\pi(x,m)$ when there is no danger of confusion about the policy we are referring to, for example, in Lemmata 3.1.1, to 3.1.3 $w(x,m)$ will denote the value function of the *equalizing policy*, in Lemmata 3.2.1, to 3.2.3 it denotes the value function of the K – *equalizing* policy, etc..

LEMMA 3.1.1. For the value function of the equalizing policy, the following relations hold for all permissible permutations of a state x,

$$w(x;m) = w(\varpi(x);m) \quad for\ m = 0, 1, \ldots, n. \tag{16}$$

PROOF. The proof is by finite induction. We first note that, since $M_\phi(x) = M_\phi(\varpi(x))$,

$$w(x;0) = w(\varpi(x);0) \tag{17}$$

Also, since $\sum_{l=1}^{N} x_l = \sum_{l=1}^{N} x_{\varpi^{-1}(l)}$ we get

$$\mu(x) = \mu(\varpi(x)) \tag{18}$$

We next observe that:

$(-1_k, \varpi(x)) = (x_{\varpi^{-1}(1)}, x_{\varpi^{-1}(2)}, \ldots, x_{\varpi^{-1}(k-1)}, x_{\varpi^{-1}(k)} - 1, x_{\varpi^{-1}(k+1)}, \ldots, x_{\varpi^{-1}(N)})$
$= \varpi(-1_{\varpi^{-1}(k)}, x)$. Assuming that the lemma holds up to $m-1$ included, we get

$$w(-1_k, \varpi(x); m-1) = w(\varpi(-1_{\varpi^{-1}(k)}, x); m-1) = w(-1_{\varpi^{-1}(k)}, x; m-1) . \tag{19}$$

Since ϖ is a permutation

$$\sum_{l=1}^{N} x_{\varpi^{-1}(l)}\, u(-1_{\varpi^{-1}(l)}, x; m-1) = \sum_{l=1}^{N} x_l\, u(-1_l, x; m-1) \tag{20}$$

In a similar to (19) way, one may show that

$$u(1_\alpha, \varpi(x); m-1) = u(\varpi(1_\alpha, x); m-1) = u(1_\alpha, x; m-1)\ . \tag{21}$$

But then, equations $(2.12)_m$,and (3) to (7) , give the result, i.e. $u(x;m) = u(\varpi(x);m)$.

We next prove the following.

<u>**LEMMA 3.1.2.**</u> $\forall\ x \in S:\ x_j \le x_i,\ C_j^0(x) \ne \emptyset,\ C_i^0(x) \ne \emptyset$ the following inequalities are true

(a) $$u(1_j, x; m) \ge u(1_i, x; m) \tag*{$(22)_m$}$$

(b) $$u(1_{a(1_j x)}, 1_j, x; m) \ge u(1_{a(1_i x)}, 1_i, x; m). \tag*{$(23)_m$}$$

<u>**PROOF.**</u> If $x_j = x_i$ the result is trivial. Hence assume $x_j < x_i$. The proof is by induction. For $m=0$, by considering cases for $M_\phi(x)$, it is easy to check that $(22)_0$ and $(23)_0$ are true. Now, let us assume they are true up to $m-1$ included. To show $(22)_m$, we first observe that

$$\mu(1_j, x) = \mu(1_i, x) \tag{24}$$

Also, since $x_j < x_i$, $x_j \le x_i - 1$ and therefore, $(22)_{m-1}$ for the state $(-1_i, x)$ implies

$$u(1_j, -1_i, x; m-1) \ge u(x; m-1) \tag*{$(25)_{m-1}$}$$

If $x_j \ne 0$, in a similar way for the state $(-1_j, x)$, we get

$$u(x; m-1) \ge u(1_i, -1_j, x; m-1) \tag*{$(26)_{m-1}$}$$

Now, whether $x_j = 0$ or not, equations $(25)_{m-1}$ and $(26)_{m-1}$ lead to

$$(x_i - x_j)\, u(x; m-1) + x_j\, u(-1_j, 1_i, x; m-1) \le$$

$$(x_i - x_j)\, u(-1_i, 1_j, x; m-1) + x_j\, u(-1_i, 1_j, x; m-1) \tag*{$(27)_{m-1}$}$$

which in turn, together with $(22)_{m-1}$ for the state $(-1_l, x)$, gives

$$\mu \sum_{\substack{l=1 \\ l \ne i,j}}^{N} x_l\, u(-1_l, 1_j, x; m-1) + \mu(x_j + 1)\, u(x; m-1) + \mu x_i\, u(-1_i, 1_j, x; m-1)$$

$$\ge \mu \sum_{\substack{l=1 \\ l \ne i,j}}^{N} x_l\, u(-1_l, 1_i, x; m-1) + \mu(x_i + 1)\, u(x; m-1) + \mu x_j\, u(-1_j, 1_i, x; m-1) \tag*{$(28)_{m-1}$}$$

But then, equations $(23)_{m-1}$, and $(28)_{m-1}$ establish $(22)_m$ (notice that Lemma 3.1.1 is also used).

To show $(23)_m$, we first notice that $x_{a(1_j x)} \geq x_{a(1_i x)}$, and then we consider cases.

Case 1: $x_{a(1_j x)} = x_{a(1_i x)}$. Then, $w(1_{a(1_j x)}, 1_j, x; m) = w(1_{a(1_i x)}, 1_j, x; m)$ by Lemma 3.1.1, and hence $(23)_m$ follows from $(22)_m$.

Case 2: $x_{a(1_j x)} > x_{a(1_i x)}$. Then we necessarily have $x_{a(1_i x)} = x_j$, and hence, by Lemma 3.1.1, $w(1_{a(1_i x)}, 1_i, x; m) = w(1_j, 1_i, x; m)$. Then $(23)_m$ becomes $w(1_{a(1_j x)}, 1_j, x; m) \geq w(1_j, 1_i, x; m)$. This is true by $(22)_m$, since, for this case, $x_{a(1_j x)} = x_j + 1 \leq x_i$. Therefore, the proof is complete.

REMARK 3.1. (a) Lemma 3.1.1 and part (a) of Lemma 3.1.2 establish the optimality of the equalizing policy since they suggest that: (i) at stage $m+1$, it is better to assign the repairman to the subsystem with the least amount of working components and (ii) if there are more than one subsystems having the same (minimum) number of working components, then, it is irrelevant to which one among these subsystems, the repairman is assigned.

(b) We make two observations about the value of $(\Pi_{n,k})$, which may be proved directly from $(2.11)_m$ using induction.

(i) For $k=0$, $v(x;m) \equiv 1$ as expected, and,

(ii) For $k=1$, $v(x;m) \equiv v(\sum_{\ell=1}^{N} x_\ell; m)$, and hence, all policies are optimal in $(\Pi_{n,1})$.

(c) If the structure of the system is the trivial $[1 \mid N; (1 \mid N_j)_{j=1,\ldots,N}]$, i.e., everything is connected in parallel, all policies are clearly the same with respect to the total expected discounted operation time of the system or its availability. This corresponds to the case of $k=1$, above and (2.6) or (15) below is an equality then.

We next state the following lemma which establishes the uniqueness of the form of the optimal policies.

LEMMA 3.1.3. *There exists $m_0 \geq 0$ such that, for $k > 1$,* inequality $(22)_m$ *is strict, i.e.,*

$\forall x \in S: x_j < x_i, C_j^0(x) \neq \emptyset, C_i^0(x) \neq \emptyset, \exists m_0 \geq 0 : w(1_j, x; m) > w(1_i, x; m)$

$\forall m \geq m_0$. (29)

PROOF. See Appendix A.

We next consider the $[\mathrm{N} \mid \mathrm{N}; (\mathrm{K}' \mid \mathrm{N}_i')_{i=1,\ldots,N}]$ system. Note that given a policy π, the structure of the system implies that the random process $Z_\pi(t; x)$, which describes the operational status of the system (reliability) at time t if the state at time 0 was x, is given by

$$Z_\pi(t;x) = \begin{cases} 1 & \text{if } M_\phi(x_\pi(t)) = \mathrm{N} \\ 0 & \text{if } M_\phi(x_\pi(t)) < \mathrm{N} \end{cases}, \tag{30}$$

where, $M_\phi(x_\pi(t)) = |\{\mathrm{i} : x_{i\pi}(t) \geq \mathrm{K}'\}|$. We next establish the following.

PROPOSITION 3.2. (a) *The* K' *− equalizing policy maximizes the probability* $p(t,x) = \mathrm{P}(Z_\pi(t;x) = 1)$ at any time instant t. (b) Any policy that is not K' − equalizing is strictly worse than the equalizing policy.

PROOF. For part (a) we establish $(2.13)_m$, $m = 0, 1, \ldots, n-1$. For $m = 0$, the proof is obvious. The induction is completed using (2.13) and (2.12) in Lemmata 3.2.1, 3.2.2 below. The proof of (b) is given by Lemma 3.2.3 below.

We need the following notation. For any state x define $L(x, \mathrm{K}') := \{l : x_l \leq \mathrm{K}'\}$.

LEMMA 3.2.1. (a) *The value remains constant over all permissible permutations of a state* x, *i.e.*

$$u(x; m) = u(\varpi(x); m) \quad \textit{for } m = 0, 1, \ldots, n. \tag{31}$$

(b) *For any two states* x *and* y *for which* (i) $L(x, K') = L(y, K')$, (ii) $\sum_{x_l \leq K'} x_l = \sum_{y_l \leq K'} y_l$ *and* (iii) $x_l = y_l \ \forall l \notin L(x, K')$, *we have*

$$u(x; m) = u(y; m) \quad \text{for } m = 0, 1, \ldots, n. \tag{32}$$

PROOF. The proof of part (a) is omitted as it is similar to that of Lemma 3.1.1 with the only addition that it also uses part (b). For the proof of part (b), let $L(K') := L(x, K')$ $(= L(y, K'))$. The statement is then trivial if $x_l = y_l = K'$ for all $l \in L(K')$. Hence, take i and j such that $x_i < K'$ and $y_j < K'$. Then, states $(1_{a_{K'}(x)}, x)$ and

$(1_{a_{K'}(y)}, y)$ satisfy assumptions (i), (ii), and (iii). Also, if $l \notin L(K')$, $(-1_l, x)$ and $(-1_l, y)$ certainly satisfy (i), (ii), and (iii). Finally, for any $l, l' \in L(K')$ $(-1_l, x)$ and $(-1_{l'}, y)$ again satisfy (i), (ii), and (iii). Part (b) follows then by using $(2.12)_m$ and finite induction.

LEMMA 3.2.2. $\forall x \in S:\ x_j \le x_i\ ,\ C_j^0(x) \neq \emptyset,\ C_i^0(x) \neq \emptyset$ *the following inequalities are true*

(a) $$w(1_j, x; m) \ge w(1_i, x; m) \qquad (33)_m$$

(b) $$w(1_{a_{K'}(1_j x)}, 1_j, x; m) \ge w(1_{a_{K'}(1_i x)}, 1_i, x; m)\,. \qquad (34)_m$$

PROOF. The proof is along exactly the same lines as the proof of Lemma 3.1.2. The only difference is that in proving $(34)_m$ and after assuming, without loss of generality, that $x_j < x_i$, one has to consider three cases.

Case (i): If $x_i < K'$, then $x_j < K'$ and $(34)_m$ holds with equality because of Lemma 3.2.1.

Case (ii): If $x_i \ge K'$ but $x_j < K'$, then

(a) If $x_j = K'-1$ and $1_j x$ is non-functioning, then there exists $k \neq i,j$ with $x_k < K'$. Then, at step m we may choose $a_{K'}(1_i x) = k$. Because of Lemma 3.2.1, we then have

$$w(1_{a_{K'}(1_j x)}, 1_j, x; m) = w(1_k, 1_j, x; m) \qquad (35)$$

$(33)_m$ then gives

$$w(1_k, 1_j, x; m) \ge w(1_k, 1_i, x; m) \qquad (36)$$

(35) and (36) establish $(34)_m$.

(b) If $x_j = K'-1$ and $1_j x$ is functioning, then $a_{K'}(1_i x) = j$ while we may also choose $a_{K'}(1_j x) = j$. Hence, using $(33)_m$ for the state $(1_j, x)$ we are led to

$$w(1_{a_{K'}(1_j x)}, 1_j, x; m) = w(1_j, 1_j, x; m) \ge w(1_i, 1_j, x; m) = w(1_{a_{K'}(1_i x)}, 1_i, x; m) \qquad (37)$$

(c) If $x_j \le K'-2$, then both $(1_j, x)$ and $(1_i, x)$ are non functioning and hence, we may choose $a_{K'}(1_i x) = a_{K'}(1_j x) = j$ which establishes $(34)_m$ through (23). Finally,

Case (iii): If $x_i \ge K'$ and $x_j \ge K'$, then

(a) If x is not functioning, then both $(1_j, x)$ and $(1_i, x)$ are non functioning. Hence, we may choose $k \neq i,j$ with $x_k < K'$, so that $a_{K'}(1_j x) = a_{K'}(1_i x) = k$. (35) and (36) are then valid and $(34)_m$ follows.

(b) If x is functioning, then $(1_j,x)$ and $(1_i,x)$ are functioning and hence, $a_{K'}(1_j x)=a(1_j x)$ and $a_{K'}(1_i x)=a(1_i x)$. The proof then is as in Lemma 3.1.2(b).

REMARK 3.2. (a) As in Proposition 3.1 , Lemma 3.2.1 and part (a) of Lemma 3.2.2 establish the optimality of the K' – equalizing policy since they suggest that at stage $m+1$, (i) if the system is down, prefer not to assign the repairman to an operational subsystem and then it is irrelevant to which non-functioning subsystem the repairman is assigned, but otherwise, (ii) it is better to assign the repairman to the subsystem with the least amount of working components while, (iii) if there are more than one subsystems having the same (minimum) number of working components, then, it is irrelevant to which, among these subsystems, the repairman is assigned. Hence, $(2.13)_{m+1}$ has been established.

(b) If the system is N-out-of-N and all subsystems are N'-out-of-N' , i.e., everything is connected in series, then all policies are clearly N'-equalizing and optimal.

We next state the following lemma which establishes the uniqueness of the form of the optimal policies.

LEMMA 3.2.3. *For any state x in S the following inequalities hold.*

(i) If $x \in B$, then for $x_j < K'$ and $x_i \geq K'$ such that $C_j^0(x) \neq \emptyset$, $C_i^0(x) \neq \emptyset$,

$$\exists m_0 \geq 0 : w(1_j,x;m) > w(1_i,x;m) \ \forall m \geq m_0 \tag{38}$$

(i) If $x \in G$, then for $x_j < x_i$ such that $C_j^0(x) \neq \emptyset$, $C_i^0(x) \neq \emptyset$,

$$\exists m_0 \geq 0 : w(1_j,x;m) > w(1_i,x;m) \ \forall m \geq m_0 \tag{39}$$

PROOF. See appendix A.

We next consider the $[1 \mid N; (K' \mid N')_{i=1,..,N}]$ system. Given a policy π , the structure of the system implies that the random variable $Z_\pi(t;x)$, which describes the operational status of the system at time t if the state at time 0 was x , is given by

$$Z_\pi(t;x)=\begin{cases} 1 & \text{if } M_\phi(x_\pi(t)) \geq 1 \\ 0 & \text{if } M_\phi(x_\pi(t))=0 \end{cases} . \tag{40}$$

where $M_\phi(x_\pi(t))= | \{i : \ x_{i\pi}(t) \geq K'\} |$.

We next show that the property of being *inequalizing* characterizes all optimal policies.

PROPOSITION 3.3. (a) *The inequalizing policy maximizes the probability* $p(t,x)=P(Z_\pi(t;x)=1)$ *at any time instant t.* (b) *For* $K'>1$, *any policy that is not inequalizing is strictly worse than the inequalizing policy.*

PROOF. As in the previous propositions, the proof of (a) is by induction on m using (2.13) and (2.12) and is done in Lemmata 3.3.1, 3.3.2 below. The proof of (b) is given in Lemma 3.3.3 below.

LEMMA 3.3.1. *The value of the inequalizing policy remains constant over all permissible permutations of a state x, i.e.*

$$w(x;n)=w(\varpi(x);n) \text{ for } m=0,1,\ldots,n. \tag{41}$$

PROOF. The result is easy to see.

LEMMA 3.3.2. $\forall x\in S:\ x_j\le x_i,\ C_j^0(x)\neq\emptyset,\ C_i^0(x)\neq\emptyset$ *the following inequalities are true*

(a) $$w(1_j,x;m)\le w(1_i,x;m) \qquad (42)_m$$

(b) $$w(1_{b(1_j x)},1_j,x;m)\le w(1_{b(1_i x)},1_i,x;m) \qquad (43)_m$$

PROOF. The proof of (a) is identical to the proof of Lemma 3.1.2 (a) with the sign of all numbered inequalities reversed. To prove part (b), we consider two cases,

(i) If $x_i<x_{b(x)}$, then Lemma 3.3.1 and $(42)_m$ establish $(43)_m$

(ii) If $x_i=x_{b(x)}$, then

$$w(1_{b(1_j x)},1_j,x;m)=w(1_i,1_j,x;m) \tag{44}$$

Also then, $x_{b(1_i x)}\ge x_j$. So, $(43)_m$ follows by using $(42)_m$ on (44) for the state $(1_i,x)$.

REMARK 3.3.1. (a) Lemma 3.3.1 and part (a) of Lemma 3.3.2 establish the optimality of the inequalizing policy with the usual argument.

(b) If $K'=1$, remark 3.1 (b) is valid and all policies are optimal.

We next state the following.

LEMMA 3.3.3. For any state x in S such that $x_j<x_i$, $C_j^0(x)\neq\emptyset$, $C_i^0(x)\neq\emptyset$ the following is true

$$\exists m_0 \geq 0 : u(1_j, x; m) < u(1_i, x; m) \; \forall m \geq m_0 \tag{45}$$

PROOF. See Appendix A.

We next consider the $[K \mid N ; (N' \mid N')_{i=1,\dots,N}]$ system. Given a policy π, the structure of the system implies that the random variable $Z_\pi(t;x)$, which describes the reliability of the system at time t if the state at time 0 was x, is given by

$$Z_\pi(t;x) = \begin{cases} 1 & \text{if } M_\phi(x_\pi(t)) \geq K \\ 0 & \text{otherwise} \end{cases}, \tag{46}$$

where $M_\phi(x_\pi(t)) = |\{i : x_{i\pi}(t) = N'\}|$. The main result is the following.

PROPOSITION 3.4. (a) *The K – inequalizing policy maximizes the probability $p(t,x) = P(Z_\pi(t;x) = 1)$ at any time instant t. (b) For $K > 1$, any policy that is not K – inequalizing is strictly worse than the equalizing policy.*

PROOF. As in the previous propositions, the proof of (a) is by induction on m using (2.13), (2.12) and is done in Lemmata 3.4.1, 3.4.2 below. The proof of (b) is given in Lemma 3.4.3 below.

To state the next lemma we need the following notation. Let $M(x,K)$ be the set of the first K subsystems in order of the number of their functioning components.

LEMMA 3.4.1. (a) *The value of the K – inequalizing policy remains constant over all permissible permutations of a state x, i.e.*

$$u(x;m) = u(\varpi(x);m) \;\; \text{for } m = 0, 1, \dots, n. \tag{47}$$

(b) *For any two states x and y for which* (i) $x_l = 0 \; \forall l \notin M(x,K)$, $y_l = 0 \; \forall l \notin M(y,K)$ and (ii) $\sum_{l \in M(x,K)} x_l = \sum_{l \in M(y,K)} y_l \leq KN'$, we have $u(x;m) = u(y;m)$ for $m = 0, 1, \dots, n$. (48)

PROOF. Part (a) is obvious. Part (b) is also obvious if (ii) is an equality. Thus, assume that both x and y are non functioning states. Then, states $(1_{b_K(x)}, x)$ and $(1_{b_K(y)}, y)$ satisfy assumptions (i) and (ii). Also, for any $l \in M(x,K)$, $l' \in M(y,K)$ such that $x_l > 0$, $y_{l'} > 0$, $(-1_l, x)$ and $(-1_{l'}, y)$ again satisfy (i) and (ii). Part (b) follows then by using $(12)_m$ and finite induction.

LEMMA 3.4.2. $\forall x \in S : x_j \leq x_i$, $C_j^0(x) \neq \emptyset$, $C_i^0(x) \neq \emptyset$ *the following inequalities are true*

$$\exists m_0 \geq 0 : u(1_j, x; m) < u(1_i, x; m) \; \forall m \geq m_0 \tag{45}$$

PROOF. See Appendix A.

We next consider the $[K \mid N; (N' \mid N')_{i=1,\dots,N}]$ system. Given a policy π, the structure of the system implies that the random variable $Z_\pi(t;x)$, which describes the reliability of the system at time t if the state at time 0 was x, is given by

$$Z_\pi(t;x) = \begin{cases} 1 & \text{if } M_\phi(x_\pi(t)) \geq K \\ 0 & \text{otherwise} \end{cases}, \tag{46}$$

where $M_\phi(x_\pi(t)) = |\{i : x_{i\pi}(t) = N'\}|$. The main result is the following.

PROPOSITION 3.4. (a) *The* K − *inequalizing policy maximizes the probability* $p(t,x) = P(Z_\pi(t;x) = 1)$ *at any time instant* t. (*b*) *For* $K > 1$, *any policy that is not* K − *inequalizing is strictly worse than the equalizing policy.*

PROOF. As in the previous propositions, the proof of (a) is by induction on m using (2.13), (2.12) and is done in Lemmata 3.4.1, 3.4.2 below. The proof of (b) is given in Lemma 3.4.3 below.

To state the next lemma we need the following notation. Let $M(x,K)$ be the set of the first K subsystems in order of the number of their functioning components.

LEMMA 3.4.1. (a) *The value of the* K − *inequalizing policy remains constant over all permissible permutations of a state* x, *i.e.*

$$u(x;m) = u(\varpi(x);m) \text{ for } m = 0, 1, \dots, n. \tag{47}$$

(b) *For any two states* x *and* y *for which* (i) $x_l = 0 \; \forall l \notin M(x,K)$, $y_l = 0 \; \forall l \notin M(y,K)$ and (ii) $\sum_{l \in M(x,K)} x_l = \sum_{l \in M(y,K)} y_l \leq KN'$, we have $u(x;m) = u(y;m)$ for $m = 0, 1, \dots, n$. (48)

PROOF. Part (a) is obvious. Part (b) is also obvious if (ii) is an equality. Thus, assume that both x and y are non functioning states. Then, states $(1_{b_K(x)}, x)$ and $(1_{b_K(y)}, y)$ satisfy assumptions (i) and (ii). Also, for any $l \in M(x,K)$, $l' \in M(y,K)$ such that $x_l > 0$, $y_{l'} > 0$, $(-1_l, x)$ and $(-1_{l'}, y)$ again satisfy (i) and (ii). Part (b) follows then by using $(12)_m$ and finite induction.

LEMMA 3.4.2. $\forall x \in S : \; x_j \leq x_i, \; C_j^0(x) \neq \emptyset, \; C_i^0(x) \neq \emptyset$ *the following inequalities are true*

(a) $$w(1_j, x; m) \leq w(1_i, x; m) \qquad (49)_m$$

(b) $$w(1_{b_K(1_j x)}, 1_j, x; m) \geq w(1_{b_K(1_i x)}, 1_i, x; m) \qquad (50)_m$$

PROOF. $(35)_m$ is proved in the same way as $(42)_m$. Because of part (b) of the previous lemma, to show $(36)_m$, it is sufficient to show $w(1_{b(1_j x)}, 1_j, x; m) \leq w(1_{b(1_i x)}, 1_i, x; m)$. This is done in a way similar to that establishing $(43)_m$.

REMARK 3.4.1 (a) Lemmata 3.4.1 and 3.4.2 (a) prove part (a) of Proposition 3.4 since they suggest that at stage $m+1$, (i) if the system is down and there are less than K subsystems with functioning components, then it is irrelevant to which non functioning subsystem the repairman is assigned, while, (ii) if the system is down and there are exactly K subsystems with functioning components, then prefer not to assign the repairman to a subsystem with no functioning components (and then it doesn't matter to which non functioning subsystem she is assigned), but otherwise, (iii) it is better to assign the repairman to the non functioning subsystem with the largest amount of working components and then (iv) if there are more than one such subsystems having the same (maximum) number of working components, then, it is irrelevant to which, among these subsystems, the repairman is assigned. Hence, $(2.13)_{m+1}$ has been established.

(b) If K=N, i.e. everything is connected in series, then all policies are clearly K – inequalizing and optimal.

The proof of part (b) of Proposition 3.4 is given by the next lemma.

LEMMA 3.4.3. (i) $\forall x \in B : \ |P(x)| = K$, *and* $x_j = 0$ *and* $0 < x_i < N'$,

$$\exists m_0 \geq 0 : w(1_j, x; m) < w(1_i, x; m) \ \forall m \geq m_0 \qquad (51)$$

(ii) $\forall x \in S : \ |P(x)| > K$, *and for* $x_j < x_i$ *such that* $C_j^0(x) \neq \emptyset$, $C_i^0(x) \neq \emptyset$,

$$\exists m_0 \geq 0 : w(1_j, x; m) < w(1_i, x; m) \ \forall m \geq m_0 \qquad (52)$$

PROOF. See Appendix A.

4. CONTROL OF ARRIVALS IN QUEUES.

In this section we establish results (1.a) to (2.b) introduced in §1. Recall that the Markovian decision processes formulation of §2 remains valid with the only difference being that of interpretation. A decision is now a decision about an arrival assignment and the components of the state vector x, x_i now denoting the number of customers at server i. We need the following notation. Let $\overline{N}_\pi(t,x)$ denote the number of idle servers at time t, given the initial state x when policy π is used. Also we define

$$p_\pi^{(1)}(t,x)=P(\{\exists i: \ x_{i\pi}(t) < K\} \mid x_\pi(0)=x) \tag{53}$$

$$p_\pi^{(2)}(t,x)=P(\{x_{i\pi}(t) < K, \ \forall i=1,\ldots,N\} \mid x_\pi(0)=x) \tag{54}$$

$$p_\pi^{(3)}(t,x)=P(\{ |i: \ x_{i\pi}(t)=N'| < K \ |\} \mid x_\pi(0)=x) \tag{55}$$

Note that $p_\pi^{(i)}(t,x)$ denotes the *probability there is at least one server with less than* K, $0 < K < N_i'$, customers (for i=1), the *probability that all servers have less than* K, $0 < K < N'$, *customers* (for i=2), *the probability that there are less than* K *servers* working at full capacity (for i=3), at time t, *given the policy* π *and the initial state* x.

The main result is the following.

Proposition 4.1. (4.1.a) *The equalizing policy minimizes stochastically* $\overline{N}_\pi(t,x)$ *at any time instant* t.

(4.1.b) *Any policy that is not equalizing is strictly worse than the equalizing policy.*

(4.2.a) *The* K − *equalizing policy minimizes the probability* $p_\pi^{(1)}(t,x)$, $\forall t$, $\forall$ K, $0 < K < N_i'$.

(4.2.b) *Any policy that is not* K − *equalizing is strictly worse than the* K − *equalizing policy.*

(4.3.a) *When* $N_i'=N'$, $\forall i$, *we have:* (a) *the inequalizing policy minimizes the probability* $p_\pi^{(2)}(t,x)$, $\forall t$, $\forall$K, $0 < K < N'$ *and* (b) *for* $K' > 1$, *any policy that is not inequalizing is strictly worse than the equalizing policy.*

(4.4.a) When $N_i'=N'$, $\forall i$, *we have:* (a) *the* K-*inequalizing policy minimizes the probability* $p_\pi^{(3)}(t,x)$, $\forall t$, $\forall$ K, $0 < K < N'$ *and* (b) *for* $K > 1$, *any policy that is not* K − *inequalizing is strictly worse than the equalizing policy.*

Proof. For case (4.1) the key idea is to observe that

$$\overline{N}_\pi(t,x)+N_\pi(t,x)=N, \ \forall t, \ x, \ \pi, \tag{56}$$

where $N_\pi(t,x)$ refers to the reward structure of the reliability problem of case (1.A). The result follows from Proposition 3.1. For case (4.2) note that

$$p_\pi^{(1)}(t,x)+p_\pi(t,x)=1, \ \forall t, \ x, \ \pi, \tag{57}$$

ıere $p_\pi(t,x)$ refers to the reward structure of the reliability problem of case (1.B). ıe result follows from Proposition 3.2. In a similar manner (4.3) and (4.4) follow ›m Propositions 3.3 and 3.4.

EMARK 4.1. The problems of maximizing the quantities that are considered in oposition 4.1 are also well defined. Using similar arguments as in §3 we conjecture at one can establish the following. The *inequalizing policy maximizes stochastically* $_t(t,x)$ *at any time instant* t. The generalization of the inequalizing policy that *ways gives priority to the server with largest* $N_i^!$ *maximizes* $p_\pi^{(1)}(t,x)$, $\forall t$, $\forall$ K, $\leq K < N_i^!$ When $N_i^! = N'$, $\forall i$, the equalizing policy *maximizes* $p_\pi^{(i)}(t,x)$, $\forall t$, $\forall$ K, $\leq K < N'$, for i=2,3.

Appendix A.

oof of LEMMA 3.1.3. The proof is by considering cases and makes use of $(2.12)_{m+1}$. , take $k > 1$ and consider first

se 1: $x_j = 0$.

bcase 1a: $M_\phi(x)=k-1$. Then, for the present case, we necessarily have $M_\phi(1_j,x)=k$ d $M_\phi(1_i,x)=k-1$. Hence, $w(1_j,x;0)=1$, $w(1_i,x;0)=0$ and, using $(2.12)_{m+1}$, we ıclude that (3.15) is true with $m_0=0$.

bcase 1b: $M_\phi(x)=k$. Then, let $x_\nu = \min\{x_l : x_l \neq 0,\ l \neq i\}$. x_ν is well defined since > 0, $M_\phi(x)=k$ and $k > 1$. Then the claim is that $w(1_j,x;x_\nu) > w(1_i,x;x_\nu)$ and hence 15) is true (by $(2.12)_{m+1}$) with $m_0 = x_\nu$. The proof of this claim is by induction on For $x_\nu = 1$, $w(1_j,-1_\nu,x;0)=1$, since $M_\phi(1_j,-1_\nu,x)=k$, and $w(1_i,-1_\nu,x;0)=0$, since $_\phi(1_i,-1_\nu,x)=k-1$. Hence, from the backwards part of $(2.12)_1$, $w(1_j,x;1) > w(1_i,x;1)$. sume the claim is true up to $x_\nu=\rho$. Then, for $x_\nu=\rho+1$ we have $(1_j,-1_\nu,x;\rho) > w(1_i,-1_\nu,x;\rho)$ and hence, using $(2.12)_{\rho+1}$, $(1_j,x;\rho+1) > w(1_i,x;\rho+1)$.

›case 1c: $M_\phi(x)=k-r$, $r=1,2,\ldots,k-1$. (Observe that the case $M_\phi(x)=0$ is of no erest, since by Lemma 3.1.1 the equalizing policy is then indifferent to which nponent the repairman is assigned). Then we show by induction on r that for

$$M_\phi(x)=k-r,\ r=1,2,\ldots,k-1,\quad 0= x_j < x_i \textit{ implies } w(1_j,x;r-1) > w(1_i,x;r-1) \qquad (1)_r$$

which, because of equation (2.12), implies that (3.15) is valid with $m_0=r-1$. To show (1) we first note that for $r=1$ this has already been proved as subcase 1a. Hence, assume that $(1)_r$ is true for $r=1,2,\ldots,\rho$ and take x with $M_\phi(x)=k-\rho-1$. Then, there has to be at least one more system with all of its components failed except j. Thus, $x_{a(1_j,x)}=0$. Of course, $x_{a(1_i,x)}=0$ also. Now, at state $(1_{a(1_j,x)},x)$ the j^{th} system has all of its components non functioning and $M_\phi(1_{a(1_j,x)},x)=k-\rho$. Hence, $(1)_\rho$ implies

$$w(1_j,1_{a(1_j,x)},x;\rho-1) > w(1_i,1_{a(1_j,x)},x;\rho-1) \tag{2}$$

But, since $x_{a(1_j,x)}=0$ and $x_{a(1_i,x)}=0$, Lemma 3.1.1 implies

$$w(1_i,1_{a(1_j,x)},x;\rho-1)=w(1_i,1_{a(1_i,x)},x;\rho-1) \tag{3}$$

(2) and (3) show that $(3.9)_{\rho-1}$ is a strict inequality which in turn implies that $(3.8)_\rho$ is a strict inequality. This establishes $(1)_\rho$.

Subcase 1d: $M_\phi(x)=k+r$, $r=0,1,\ldots,N-k$. Then, since $k>1$, there exist at least $r+2$ systems with at least one functioning component. Hence, the following recursive definition is valid: Let $x_{\nu_1}=\min\{x_l : x_l \neq 0,\ l\neq i\}$ and, provided $r>0$, $x_{\nu_{\xi+1}}=\min\{x_l : x_l\neq 0,\ l\neq i,\ \nu_1,\ldots,\nu_\xi\}$ for $\xi=1,\ldots,r$. We show by induction on r that

For $M_\phi(x)=k+r$, $r=0,1,\ldots,N-k$, $0=x_j<x_i$ implies

$$w(1_j,x;\sum_{\xi=1}^{r+1}x_{\nu_\xi}) > w(1_i,x;\sum_{\xi=1}^{r+1}x_{\nu_\xi}) \tag*{$(4)_r$}$$

and, as in subcase 1c, this is enough to establish (3.15) with $m_0=\sum_{\xi=1}^{r+1}x_{\nu_\xi}$.

$(4)_0$ is true by subcase 1b. Assume then that (4) is true up to ρ, for $0\le\rho\le N-k-1$, and take x with $M_\phi(x)=k+\rho+1$. Then, $M_\phi(-y_{\nu_{\rho+2}},x)=k+\rho$, where $y=x_{\nu_{\rho+2}}$ and hence, $(4)_\rho$ implies

$$w(1_j,-y_{\nu_{\rho+2}},x;\sum_{\xi=1}^{\rho+1}x_{\nu_\xi}) > w(1_i,-y_{\nu_{\rho+2}},x;\sum_{\xi=1}^{\rho+1}x_{\nu_\xi}). \tag{5}$$

(5) shows that inequality $(3.14)_\zeta$ is strict, where $\zeta=\sum_{\xi=1}^{\rho+1}x_{\nu_\xi}$ and hence, $(3.8)_{\zeta+1}$ is strict, i.e.

$$w(1_j,-(y-1)_{\nu_{\rho+2}},x;\sum_{\xi=1}^{\rho+1}x_{\nu_\xi}+1) > w(1_i,-(y-1)_{\nu_{\rho+2}},x;\sum_{\xi=1}^{\rho+1}x_{\nu_\xi}+1).$$

Repeating this argument recursively we are led to

$$w(1_j,x;\sum_{\xi=1}^{\rho+2}x_{\nu_\xi}) > w(1_i,x;\sum_{\xi=1}^{\rho+2}x_{\nu_\xi}) \tag{6}$$

which completes the induction step and hence $(4)_r$ is true.

Subcases 1c and 1d show that if $x_j=0$, (3.15) is true for $m_0=\sum_{i=1}^{N}N_i+k-2$, which finishes the proof for Case 1.

Case 2: $x_j > 0$.

The proof is by induction on the number of working components of system j (i.e. x_j) at state x. (3.15) is true for $x_j=0$ by case 1 and for m_0 as above. Let us assume (3.15) is true for $x_j=\nu$ and for $m_0=\nu+\sum_{i=1}^{N} N_i+k-2$. Take then x with $x_j=\nu+1$ and consider state $(-1_j, x)$. Because of the inductive hypothesis, there exists m_0 such that for $m \geq m_0$

$$w(x;m) > w(1_i,-1_j, x;m) \tag{7}$$

Now, (7) is $(3.12)_m$ in strict form. Hence, since $x_j > 0$, $(3.13)_m$ is strict and therefore $(3.8)_{m+1}$ is strict. Hence, (3.15) is true for $m \geq m_0+1$. This completes the inductive step and shows that (3.15) is true with m_0 independent of x. In particular, we may take $m_0=2\sum_{i=1}^{N} N_i+k-2$.

__Proof of LEMMA 3.2.3.__ Because of equation (2.12) and Lemma 3.2.1, to prove (3.24) and (3.25) it is sufficient to find m_0. Strict inequality will then automatically hold for all $m \geq m_0$. To prove (3.24), we show that for a state x such that $C_j^0(x) \neq \emptyset$, $C_i^0(x) \neq \emptyset$, $M_\phi(x)=N-r$, $r=1,\ldots,N-1$, $x_j < K'$ and $x_i \geq K'$, the following inequality holds

$$w(1_j,x;rK'-\sum_{x_l < K'} x_l-1) > w(1_i,x;rK'-\sum_{x_l < K'} x_l-1) \tag{8}$$

The proof of (8) is by finite double induction on the number of failed subsystems at state x, r, and on the number of failed components necessary for the operation of subsystem j, $s:=K'-x_j$. Call the corresponding to r , s inequalities $(8)_{r,s}$ and first consider the case in which $r=1$ and $s=1$. Then $M_\phi(1_j,x)=N$, while $M_\phi(1_i,x)=N-1$. Therefore, $w(1_j,x;0) > w(1_i,x;0)$ and $(8)_{1,1}$ holds. Now, assume $(8)_{r,s}$ is true for $r=1,\ldots,\rho$ and $s=1,\ldots,\sigma$. We show it is true for $r=\rho$ and $s=\sigma+1$. Indeed, for the state $(1_j,x)$ we have $M_\phi(1_j,x)=N-\rho$ and the number of failed components necessary for the operation of subsystem j is σ. From the inductive step we conclude that

$$w(1_j,1_j,x;\rho K'-\sum_{x_l < K'} x_l-2) > w(1_j,1_i,x;\rho K'-\sum_{x_l < K'} x_l-2) \tag{9}$$

Now, we may choose $a_{K'}(1_jx)=a_{K'}(1_ix)=j$. Hence, (9) becomes

$$w(1_{a_{K'}(1_jx)},1_j,x;\rho K'-\sum_{x_l < K'} x_l-2) > w(1_{a_{K'}(1_ix)},1_j,1_i,x;\rho K'-\sum_{x_l < K'} x_l-2) \tag{10}$$

which leads to $(8)_{\rho,\sigma+1}$ by using (2.12) and the inequalities (basically those concerning the forward motion of the system) involved in the proof of Lemma 3.2.2 . The next step is to assume the validity of (8) for $r=1,\ldots,\rho$ and any s and show it is true for $r=\rho+1$ and $s=1$. So, take x with r and s as above and consider $(1_j,x)$. We have $M_\phi(1_j,x)=N-\rho$. Then, take any non functioning subsystem ν at state $(1_j,x)$. The validity of $(8)_\rho$ implies

$$w(1_\nu,1_j,x;\rho K'-\sum_{\substack{x_l<K'\\ l\neq j}} x_l-1) > w(1_i,1_j,x;\rho K'-\sum_{\substack{x_l<K'\\ l\neq j}} x_l-1) \tag{11}$$

Next, choose $a_{K'}(1_jx)=\nu$ and $a_{K'}(1_ix)=j$ to transform (11) to

$$w(1_{a_{K'}(1_jx)},1_j,x;\rho K'-\sum_{\substack{x_l<K'\\ l\neq j}} x_l-1) > w(1_{a_{K'}(1_ix)},1_i,x;\rho K'-\sum_{\substack{x_l<K'\\ l\neq j}} x_l-1) \tag{12}$$

(12), using (2.12) and the inequalities involved in the proof of Lemma 3.2.2, gives

$$w(1_j,x;\rho K'-\sum_{\substack{x_l<K'\\ l\neq j}} x_l) > w(1_i,x;\rho K'-\sum_{\substack{x_l<K'\\ l\neq j}} x_l) \tag{13}$$

But, since $x_j=K'-1$, (13) may be rewritten

$$w(1_j,x;(\rho+1)K'-\sum_{x_l<K'} x_l-1) > w(1_i,x;(\rho+1)K'-\sum_{x_l<K'} x_l-1) \tag{14}$$

With (14) the proof of (8) is complete. To prove (3.25), we show that for a state x such that $C_j^0(x)\neq\emptyset$, $C_i^0(x)\neq\emptyset$, $M_\phi(x)=N$, $x_j=K'+r$, $r=0,\ldots,N_j-K'$, the following inequality holds

$$w(1_j,x;r+1) > w(1_i,x;r+1) \tag{15}$$

The proof is by induction. Since $(-1_j,x)$ is non-functioning, for $r=0$ we get $w(x;0)=1$ and $w(-1_j,1_i,x;0)=0$. Therefore, using (2.12) and the inequalities (basically those concerning the backwards motion of the system) involved in the proof of Lemma 3.2.2, we deduce that $w(1_j,x;1) > w(1_i,x;1)$. Now assume that (50) is true for $r=0,\ldots,\rho$. Then, for $x_j=K'+\rho+1$, consider the state $(-1_j,x)$. The system is functioning and the jth subsystem has $K'+\rho$ working components at that state. Hence, from the inductive step we get $w(x;\rho+1) > w(-1_j,1_i,x;\rho+1)$ and therefore, using the same argument as before, $w(1_j,x;\rho+2) > w(1_i,x;\rho+2)$. With this, the proof of (15) and the lemma is complete.

Proof of LEMMA 3.3.3. The proof is by considering cases and makes use of $(2.12)_{m+1}$. *Case* 1: The i-th subsystem is not functioning at state x. Then, we show that m_0 may be taken to be

$$m_0= \sum_{x_l \geq K'} x_l-x_i \ -(M_\phi(x)-1)(K'-1) \tag{16}$$

The proof is by simultaneous induction on $M_\phi(x)$, x_i and x_ν, where $x_\nu:=\min\{x_l: x_l \geq K'\}$ (If $\{x_l: x_l \geq K'\}=\emptyset$, then $x_\nu=0$). So, let first $M_\phi(x)=0$, $x_\nu=0$ and $x_i=K'-1$. Then, $w(1_j,x;0)=0$, $w(1_i,x;0)=1$, and (16) is valid. Assume now that (16) is valid for $M_\phi(x)=0$ and $x_i=K'-(r-1)$, $r=2,\ldots,K'$ (of course $x_\nu=0$ then) and take x with $M_\phi(x)=0$ and $x_i=K'-r$. Then, by considering state $(1_i,x)$, we get $w(1_j,1_i,x;r-2) < w(2_i,x;r-2)$ which leads to $w(1_j,x;r-1) < w(1_i,x;r-1)$. Hence, (16) is true for $M_\phi(x)=0$ (and $x_\nu=0$). Second, let $M_\phi(x)=1$, $x_\nu=K'$ and $x_i=K'-r$, $r=1,\ldots,K'$. Then, for $(-1_\nu,x)$ we have $M_\phi(-1_\nu,x)=0$ and hence, from the first step of the inductive proof,

$$w(-1_\nu,1_j,x;K'-x_i-1) < w(-1_\nu,1_i,x;K'-x_i-1)$$

which implies $w(1_j,x;K'-x_i) < w(1_i,x;K'-x_i)$ and therefore (16) is valid. Assume now that (16) is true for $M_\phi(x)=1$, $x_\nu=K'+s-1$, $s=1,\ldots,N'-K'$, and $x_i=K'-r$, $r=1,\ldots,K'$ and consider x with $M_\phi(x)=1$, $x_\nu=K'+s$, and $x_i=K'-r$. Then, the inductive hypothesis leads to

$$w(-1_\nu,1_j,x;K'+s-1-x_i) < w(-1_\nu,1_i,x;K'+s-1-x_i)$$

and thus $w(1_j,x;K'+s-x_i) < w(1_i,x;K'+s-x_i)$. Hence, (16) is true for $M_\phi(x)=1$ and $x_i=K'-r$, $r=1,\ldots,K'$. Third, assume (16) is true for $M_\phi(x)=u$ and $x_i=K'-r$ and consider x with $M_\phi(x)=u+1$, and $x_i=K'-r$. Then, for $x_\nu=K'$, consider $(-1_\nu,x)$. This state has u functioning subsystems and hence

$$w(-1_\nu,1_j,x;\sum_{x_l \geq K'} x_l-x_\nu-x_i-(u-1)(K'-1)) < w(-1_\nu,1_i,x;\sum_{x_l \geq K'} x_l-x_\nu-x_i-(u-1)(K'-1))$$

which implies

$$w(1_j,x;\sum_{x_l \geq K'} x_l-x_\nu-x_i-(u-1)(K'-1)+1) < w(1_i,x;\sum_{x_l \geq K'} x_l-x_\nu-x_i-(u-1)(K'-1)+1)$$

But since,

$$\sum_{x_l \geq K'} x_l-x_\nu-x_i-(u-1)(K'-1)+1= \sum_{x_l \geq K'} x_l-x_i-u(K'-1)$$

we conclude that (16) is true for $M_\phi(x)=u+1$, $x_i=K'-r$ and $x_\nu=K'$. Next assume that (16) is true for $M_\phi(x)=u+1$, $x_i=K'-r$ and $x_\nu=K'+s-1$, $s=1,\ldots,N'-K'$ and consider x with $M_\phi(x)=u+1$, $x_i=K'-r$ and $x_\nu=K'+s$. Then, the νth subsystem of $(-1_\nu,x)$ has

$K'+s-1$ operating components and hence, with the usual argument,

$$w(1_j,x;\sum_{x_l\geq K'}x_l-1-x_i-u(K'-1)+1) < w(1_i,x;\sum_{x_l\geq K'}x_l-1-x_i-u(K'-1)+1)$$

which establishes (16) for $M_\phi(x)=u+1$, $x_i=K'-r$. But then, (16) is true for any $M_\phi(x)$ and arbitrary x_i, provided the ith subsystem is non functioning. This finishes Case 1.

Case 2: The i-th subsystem is functioning at state x. Then, we show that m_0 may be taken to be

$$m_0=\sum_{x_l\geq K'}x_l-M_\phi(x)(K'-1) \tag{17}$$

The proof is by simultaneous induction on $M_\phi(x)$, x_i and x_ν, where $x_\nu:=\min\{x_l: l\neq i, x_l\geq K'\}$. First, we establish (17) for $M_\phi(x)=1$ (and therefore, $x_\nu=0$). For $x_i=K'$, $(-1_i,x)$ has the ith subsystem non functioning and therefore $w(1_j,-1_i,x;0)=0$, $w(x;0)=1$, which implies $w(1_j,x;1)<w(1_i,x;1)$. Hence, (17) is valid for $x_i=K'$. Next, assume (17) is true for $x_i=K'+r-1$, $r=1,\ldots,N-K'$ and consider a state x, with $M_\phi(x)=1$, $x_i=K'+r$. Then, with the usual argument using state $(-1_i,x)$, one may show that $w(1_j,x;r+1)<w(1_i,x;r+1)$. So, (17) is true for all x such that $M_\phi(x)=1$. Second, let $M_\phi(x)=2$, x_i arbitrary, and $x_\nu=K'$. From the previous step we get $w(-1_\nu,1_j,x;x_i-K'+1)<w(-1_\nu,1_i,x;x_i-K'+1)$. Thus, $w(1_j,x;x_i-K'+2)<w(1_i,x;x_i-K'+2)$ and hence, (17) is true for $x_\nu=K'$. Assume (17) is true for $x_\nu=K'+r-1$, $r=1,\ldots,N-K'$. Then the usual argument implies that (17) is true for $x_\nu=K'+1$, which finishes the case $M_\phi(x)=2$. Third, assume that (17) is valid for $M_\phi(x)=u$, $u=2,\ldots,N-1$. By considering $(-1_\nu,x)$ for $x_\nu=K'$, $K'+1,\ldots$, it is straightforward to establish (17) for $M_\phi(x)=u+1$. This finishes the induction for Case 2 and the proof of the Lemma.

Proof of LEMMA 3.4.3. To prove (3.37), we show by backward induction on $U(x)=\sum_{l\in P(x)}x_l$ that we may choose

$$m_0=KN'-1-U(x) \tag{18}$$

So let $U(x)=KN'-1$. Then, $w(1_j,x;0)=0$, $w(1_i,x;0)=1$, and (18) is true. Assume (18) is true for $U(x)=KN'-r+1$ and take x with $U(x)=KN'-r$. Then, $U(1_{b_K(1_j x)},x)=KN'-r+1$ and hence

$$w(1_j,1_{b_K(1_j x)},x;KN'-r+1)\geq w(1_i,1_{b_K(1_j x)},x;KN'-r+1)=w(1_i,1_{b_K(1_i x)},x;KN'-r+1)$$

which implies the result for $U(x)=KN'-r$. To prove (3.38), we consider cases.

Case (1): $M_\phi(x)=K-1$.

Let $x_\nu=\max\{x_l:\ x_l < N'\}$. Using induction on both x_i and x_ν, we show that we may choose

$$m_0=N'-x_i-1+\sum_{\theta=x_i+1}^{x_\nu} |\{x_l:\ x_l=\theta\ \}| \tag{19}$$

Let first $x_i=N'-1$. Then, $w(1_j,x;0)=0$, $w(1_i,x;0)=1$, and (19) is true. Assume (19) is true for $x_i=N'-r+1$ and take x with $x_i=N'-r$. Then, x_ν takes values in $\{x_i, x_i+1,\ldots,x_i+(N'-x_i-1)\}$. If $x_\nu=x_i$, consider $(1_i,x)$. This state has $N'-r+1$ functioning components at the ith subsystem and hence, using the induction step, after noticing that for this case $m_0=N'-(x_i+1)-1=r-2$, we conclude that $w(1_j,1_i,x;r-2) < w(1_i,1_i,x;r-2)$. This leads to: $w(1_j,1_{b_K(1_j x)},x;r-2) <$ $w(1_i,1_{b_K(1_i x)},x;r-2)$, hence $w(1_j,x;r-1) < w(1_i,x;r-1)$ and thus (19) is established for $x_i=N'-r$ and $x_\nu=x_i$. Assume now that (19) is true for $x_i=N'-r$ and $x_\nu=x_i+u-1$ for $u=1,\ldots,N-x_i-1$. Then, take x with $x_i=N'-r$ and $x_\nu=x_i+u$ and consider the state y which is defined by

$$y_l=\begin{cases} x_l & \text{if } x_l < x_\nu \\ x_\nu-1 & \text{if } x_l=x_\nu \end{cases}$$

Then, from the inductive assumption we deduce that

$$w(1_j,y;N'-x_i-1+\sum_{\theta=x_i+1}^{x_\nu-1} |\{x_l:\ x_l=\theta\ \}|) < w(1_i,y;N'-x_i-1+\sum_{\theta=x_i+1}^{x_\nu-1} |\{x_l:\ x_l=\theta\ \}|)$$

which implies

$$w(1_j,x;N'-x_i-1+\sum_{\theta=x_i+1}^{x_\nu} |\{x_l:\ x_l=\theta\ \}|) < w(1_i,x;N'-x_i-1+\sum_{\theta=x_i+1}^{x_\nu} |\{x_l:\ x_l=\theta\ \}|)$$

This finishes the proof for this case.

Case (2) $M_\phi(x)\geq K$.

Then, we use induction on $M_\phi(x)$ to show that m_0 may be chosen by

$$m_0=N'-x_i+\sum_{\theta=x_i+1}^{N_i-1} |\{x_l:\ x_l=\theta\ \}|+M_\phi(x)-K \tag{20}$$

Take first $M_\phi(x)=K$ and let ν be any functioning subsystem. Then, $M_\phi(-1_\nu,x)=K-1$ and (19) implies:

$$w(1_j,-1_\nu,x;N'-x_i-1+\sum_{\theta=x_i+1}^{N_i-1} |\{x_l:\ x_l=\theta\ \}|) <$$
$$w(1_i,-1_\nu,x;N'-x_i-1+\sum_{\theta=x_i+1}^{N_i-1} |\{x_l:\ x_l=\theta\ \}|)\ ,$$

which is sufficient to prove (20) for $M_\phi(x)=K$. Next, assume the validity of (20) up to $M_\phi(x)=K+r-1$. By examining $(-1_\nu,x)$, where ν is any functioning subsystem, it is straightforward then to check the validity of (20) for $M_\phi(x)=K+r$.

Case (3) $M_\phi(x) < K-1$. Let then $P_k(x)$ represent the set of the k first subsystems in order of the number of their functioning components. Since we are now dealing with (3.38), $P_k(x) \subset P(x)$. For this case we consider three subcases.

Subcase (3a): $x_j=0$ and $i \in P_k(x)$. Then define $U_k(x) := \sum_{l \in P_k(x)} x_l$. Using an approach similar to the one that proved (18), one may then show that it is sufficient to choose $m_0=KN^r-1-U_k(x)$.

Subcase (3b): $x_j=0$ and $i \notin P_k(x)$. To proceed we need the following. We order all subsystems according to the number of their functioning components and we denote by $<l>$ the order of the lth subsystem (e.g., by $<5>=1$, we mean that the 5th subsystem has the largest amount of functioning components). Then, we consider the state y defined by

$$y_l=\begin{cases} x_l & \text{if } <l> < K \\ x_l-(x_l-x_i) & \text{if } K \le <l> \le <i> \\ x_l & \text{otherwise} \end{cases} .$$

State y satisfies the assumptions of subcase (3a), hence,

$$w(1_j,y;KN^r-1-U_k(y)) > w(1_i,y;KN^r-1-U_k(y))$$

which in turn implies the validity of (3.38) with $m_0=KN^r-1-U_k(y)+\sum_{<l>=K}^{<i>}(x_l-x_i)$.

Subcase (3c): $x_j > 0$. Then, consider the state $z=(-x_j,x)$. The relative order of i is the same in x and z, and z satisfies the assumptions of either subcase (3a) or subcase (3b). Hence,

$$w(1_j,x;m_0) < w(1_i,x;m_0)$$

where the present m_0 is the m_0 of the corresponding subcase increased by x_j.

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